

# The taming of the red dragon

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Spontaneous topics in theoretical physics

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*There is the broad object seen in  $\pi\pi$  scattering, often called “background”, which extends from about 400 MeV up to about 1700 MeV. This object we consider as a single broad resonance<sup>2</sup> which we identify as the lightest glueball with quantum numbers  $J^{PC} = 0^{++} \dots$*

<sup>2</sup> we refer to it as **red dragon**

P. Minkowski and W. Ochs, Eur. Phys. J. C9 (1999) 283

# Low energy pion physics

- Plays a crucial role whenever the strong interaction is involved at low energies

Example: Standard model prediction for muon magnetic moment

- Main experiments on  $\pi\pi$  scattering were done in the seventies. What's new ?

- Significant theoretical progress, based on ChPT + dispersion theory

- New precision data:

$K \rightarrow \pi\pi\ell\nu$	E865	Brookhaven
pionic atoms	DIRAC	CERN
$K \rightarrow 3\pi$	NA48/2	CERN

- Lattice results on  $M_\pi, F_\pi, a_0^2, \langle r^2 \rangle_s$

# Analyticity and crossing

- $\pi\pi$  scattering is special: crossed channels are identical
- ⇒  $\text{Re } T(s, t)$  can be represented as a twice subtracted dispersion integral over  $\text{Im } T(s, t)$  in physical region

S.M. Roy 1971

- The 2 subtraction constants can be identified with the  $S$ -wave scattering lengths:

$$a_0^0, a_0^2 \begin{array}{l} \leftarrow \text{isospin} \\ \leftarrow \text{angular momentum} \end{array}$$

- Representation leads to dispersion relations for the individual partial waves: *Roy equations*

# Roy equations

- Pioneering work on the physics of the Roy equations: Basdevant, Froggatt & Petersen 1974
- Dispersion integrals converge rapidly (2 subtractions)
- ⇒ Crude phenomenological information on  $\text{Im } T(s, t)$  for energies above 800 MeV suffices
- ⇒ Given  $a_0^0, a_0^2$ , the scattering amplitude can be calculated to within small uncertainties

Ananthanarayan, Colangelo, Gasser & L. 2001

Descotes, Fuchs, Girlanda & Stern 2002

⇒  $a_0^0, a_0^2$  are the essential parameters at low energy

- Main problem in early work:  $a_0^0, a_0^2$  poorly known  
Experimental information near threshold is meagre

# Low energy theorems

- Chiral perturbation theory provides the missing piece: theoretical prediction for  $a_0^0, a_0^2$

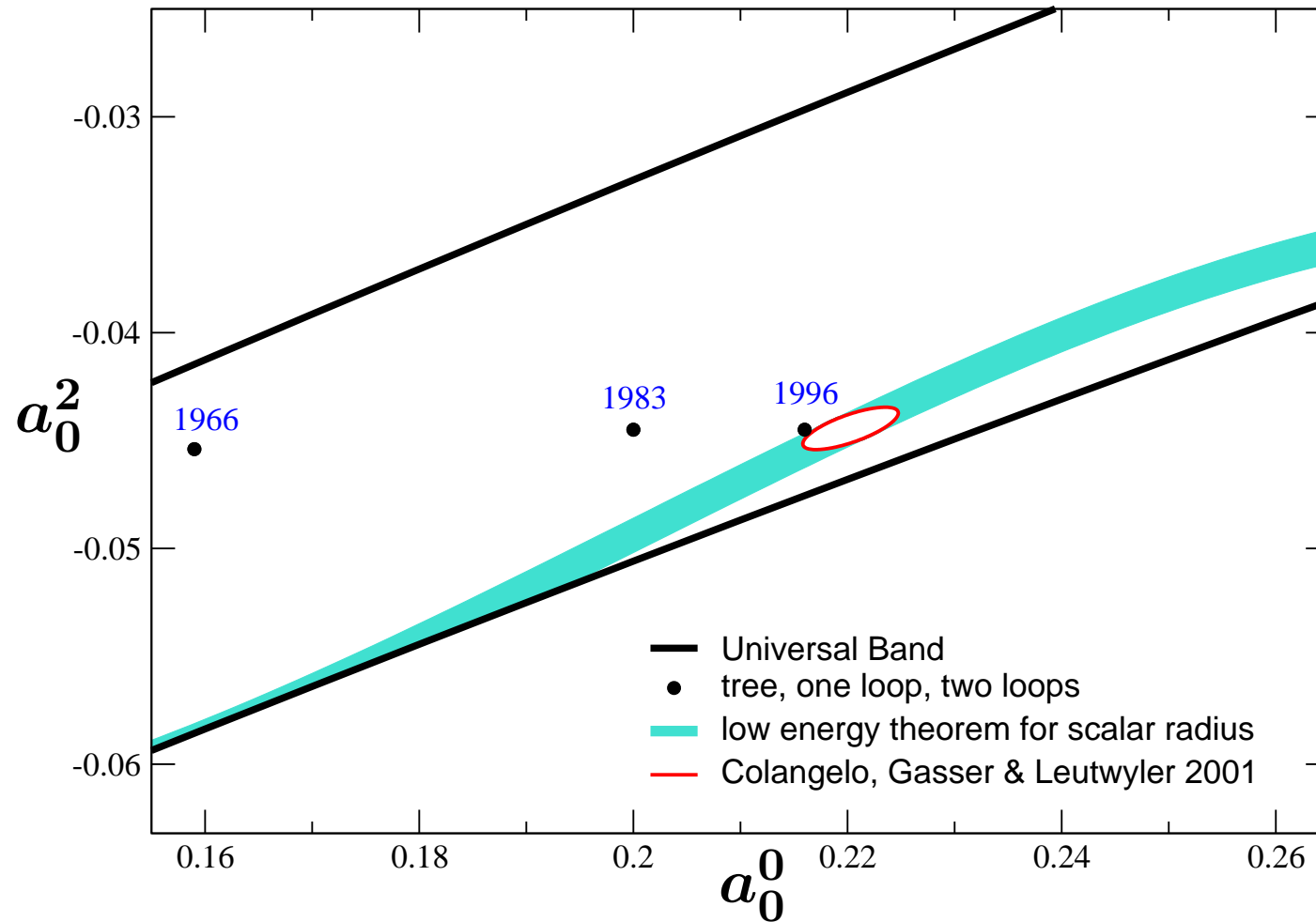
Weinberg 1966, Gasser & L. 1983, Bijmens, Colangelo, Ecker, Gasser & Sainio 1996

- Most accurate results for  $a_0^0, a_0^2$  are obtained by matching the chiral and dispersive representations near the center of the Mandelstam triangle

Colangelo, Gasser & L. 2001

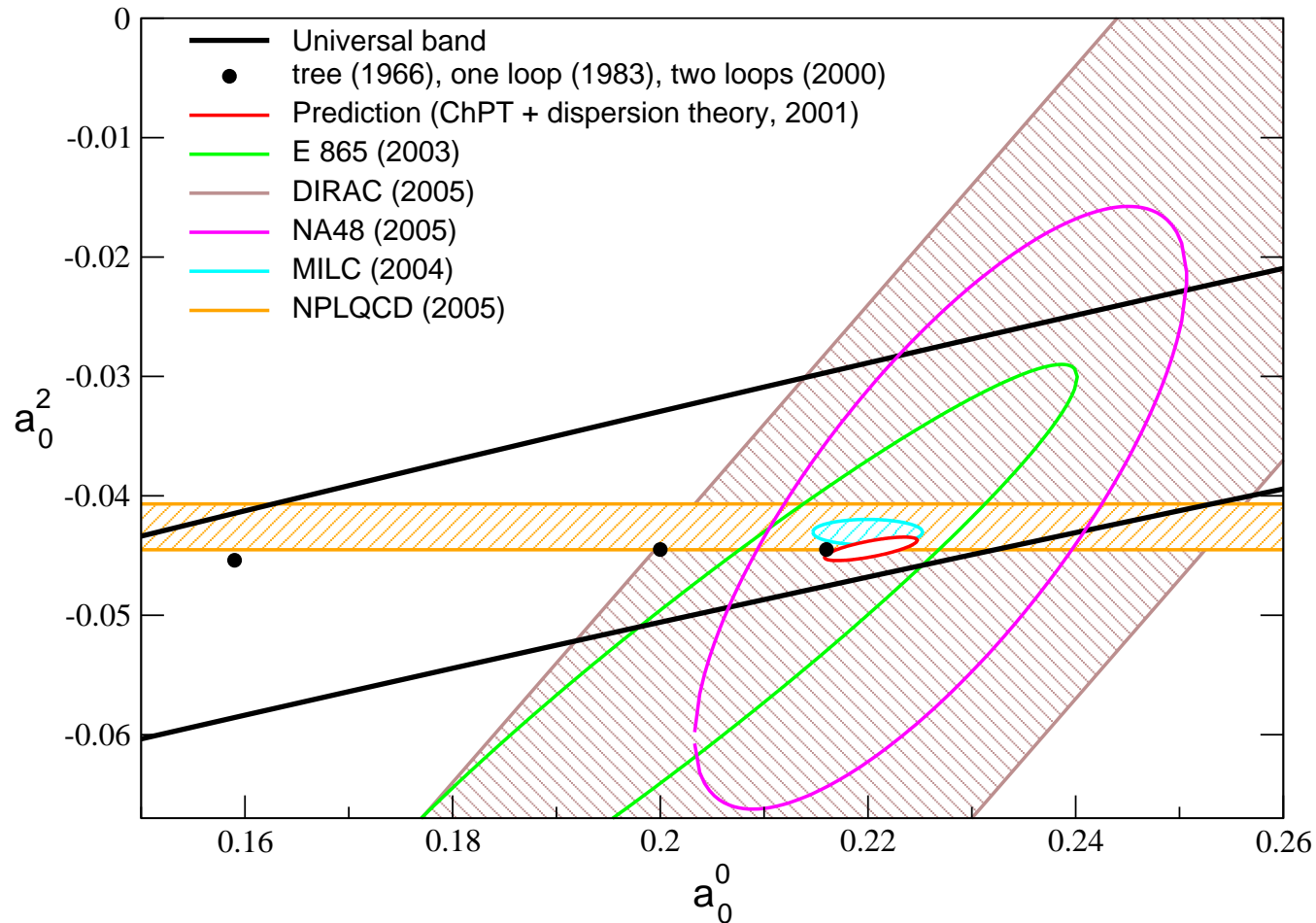
- In combination with the low energy theorems for  $a_0^0, a_0^2$ , the dispersion relations for the partial waves fix the  $\pi\pi$  scattering amplitude to an incredible degree of accuracy

# Predictions for the S-wave $\pi\pi$ scattering lengths



Sizeable corrections in  $a_0^0$ , while  $a_0^2$  nearly stays put

# Tests of the predictions for $a_0^0$ , $a_0^2$ : experiment & lattice



Theory is ahead of experiment ...



# The red dragon

I. Caprini, G. Colangelo and H. Leutwyler, Phys. Rev. Lett. 96 (2006) 132001

- Does QCD have a resonance near threshold ?
- Why care ?
  - Concerns the nonperturbative domain of QCD
  - Quark and gluon degrees of freedom useless there
  - ⇒ Understanding very poor, pattern of energy levels ?
    - Lowest resonance:  $\sigma$  ?  $\rho$  ?
- Resonance  $\leftrightarrow$  pole on second sheet
  - Poles are universal
  - Pole position is unambiguous, even if width is large
  - Where is the pole closest to the origin ?

$f_0(600)$   
or  $\sigma$

$$I^G(J^{PC}) = 0^+(0^{++})$$

A REVIEW GOES HERE – Check our WWW List of Reviews

**$f_0(600)$  T-MATRIX POLE  $\sqrt{s}$**

Note that  $\Gamma \approx 2 \text{Im}(\sqrt{s_{\text{pole}}})$ .

VALUE (MeV)	DOCUMENT ID	TECN	COMMENT
<b>(400–1200)–i(300–500) OUR ESTIMATE</b>			
● ● ● We do not use the following data for averages, fits, limits, etc. ● ● ●			
(541 ± 39)–i(252 ± 42)	1 ABLIKIM	04A BES2	$J/\psi \rightarrow \omega\pi^+\pi^-$
(528 ± 32)–i(207 ± 23)	2 GALLEGOS	04 RVUE	Compilation
(440 ± 8)–i(212 ± 15)	3 PELAEZ	04A RVUE	$\pi\pi \rightarrow \pi\pi$
(533 ± 25)–i(247 ± 25)	4 BUGG	03 RVUE	
532 – i272	BLACK	01 RVUE	$\pi^0\pi^0 \rightarrow \pi^0\pi^0$
(470 ± 30)–i(295 ± 20)	5 COLANGELO	01 RVUE	$\pi\pi \rightarrow \pi\pi$
(535 <sup>+48</sup> <sub>-36</sub> )–i(155 <sup>+76</sup> <sub>-53</sub> )	6 ISHIDA	01	$\Upsilon(3S) \rightarrow \Upsilon\pi\pi$
610 ± 14 – i620 ± 26	7 SUROVTSEV	01 RVUE	$\pi\pi \rightarrow \pi\pi, K\bar{K}$
(558 <sup>+34</sup> <sub>-27</sub> )–i(196 <sup>+32</sup> <sub>-41</sub> )	ISHIDA	00B	$p\bar{p} \rightarrow \pi^0\pi^0\pi^0$
445 – i235	HANNAH	99 RVUE	$\pi$ scalar form factor
(523 ± 12)–i(259 ± 7)	KAMINSKI	99 RVUE	$\pi\pi \rightarrow \pi\pi, K\bar{K}, \sigma\sigma$
442 – i 227	OLLER	99 RVUE	$\pi\pi \rightarrow \pi\pi, K\bar{K}$
469 – i203	OLLER	99B RVUE	$\pi\pi \rightarrow \pi\pi, K\bar{K}$
445 – i221	OLLER	99C RVUE	$\pi\pi \rightarrow \pi\pi, K\bar{K}, \eta\eta$
(1530 <sup>+90</sup> <sub>-250</sub> )–i(560 ± 40)	ANISOVICH	98B RVUE	Compilation
420 – i 212	LOCHER	98 RVUE	$\pi\pi \rightarrow \pi\pi, K\bar{K}$
(602 ± 26)–i(196 ± 27)	8 ISHIDA	97	$\pi\pi \rightarrow \pi\pi$
(537 ± 20)–i(250 ± 17)	9 KAMINSKI	97B RVUE	$\pi\pi \rightarrow \pi\pi, K\bar{K}, 4\pi$
470 – i250	10,11 TORNQVIST	96 RVUE	$\pi\pi \rightarrow \pi\pi, K\bar{K}, K\pi, \eta\pi$
~ (1100 – i300)	AMSLER	95B CBAR	$\bar{p}p \rightarrow 3\pi^0$
400 – i500	11,12 AMSLER	95D CBAR	$\bar{p}p \rightarrow 3\pi^0$
1100 – i137	11,13 AMSLER	95D CBAR	$\bar{p}p \rightarrow 3\pi^0$
387 – i305	11,14 JANSSEN	95 RVUE	$\pi\pi \rightarrow \pi\pi, K\bar{K}$
525 – i269	15 ACHASOV	94 RVUE	$\pi\pi \rightarrow \pi\pi$
(506 ± 10)–i(247 ± 3)	KAMINSKI	94 RVUE	$\pi\pi \rightarrow \pi\pi, K\bar{K}$
370 – i356	16 ZOU	94B RVUE	$\pi\pi \rightarrow \pi\pi, K\bar{K}$
408 – i342	11,16 ZOU	93 RVUE	$\pi\pi \rightarrow \pi\pi, K\bar{K}$
870 – i370	11,17 AU	87 RVUE	$\pi\pi \rightarrow \pi\pi, K\bar{K}$
470 – i208	18 BEVEREN	86 RVUE	$\pi\pi \rightarrow \pi\pi, K\bar{K}, \eta\eta, \dots$
(750 ± 50)–i(450 ± 50)	19 ESTABROOKS	79 RVUE	$\pi\pi \rightarrow \pi\pi, K\bar{K}$
(660 ± 100)–i(320 ± 70)	PROTOPOP...	73 HBC	$\pi\pi \rightarrow \pi\pi, K\bar{K}$
650 – i370	20 BASDEVANT	72 RVUE	$\pi\pi \rightarrow \pi\pi$

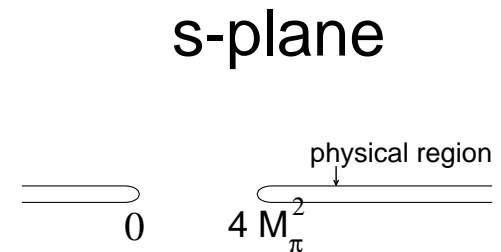
# Model independent determination of the pole

- All of the results quoted by the PDG are obtained by
  - (a) parametrizing the data for real values of  $s$
  - (b) continuing this parametrization analytically in  $s$
- ⇒ Result is sensitive to the parametrization used
- We found a model independent method:
  1. Poles on second sheet are zeros on first sheet
  2. The Roy equations are valid for complex values of  $s$ , in a limited region of the first sheet
- ⇒ Exact representation of the partial waves in terms of observable quantities, valid for complex values of  $s$
- 3. Can evaluate this representation to good precision and determine the zeros numerically

## Pole on second sheet $\leftrightarrow$ zero on first sheet

- $S_0^0(s) = \eta_0^0(s) \exp 2i\delta_0^0(s)$

$S_0^0(s)$  is analytic in the cut plane



- For  $0 < s < 4M_\pi^2$ ,  $S_0^0(s)$  is real

$\Rightarrow S_0^0(s^*) = S_0^0(s)^*$

$x$  in elastic interval:  $S_0^0(x \pm i\epsilon) = \exp \pm 2i\delta_0^0(x)$

- Second sheet is reached by continuation across the elastic interval of the right hand cut

$$S_0^0(x - i\epsilon)^{II} = S_0^0(x + i\epsilon)^I = 1/S_0^0(x - i\epsilon)^I$$

Analyticity  $\Rightarrow$   $S_0^0(s)^{II} = 1/S_0^0(s)^I$  valid  $\forall s$

Pole in  $S_0^0(s)^{II} \iff$  zero in  $S_0^0(s)^I$

## Roy equation for the isoscalar $S$ -wave

$$S_0^0(s) = 1 + 2i\rho t_0^0(s) \quad \rho = \sqrt{1 - 4M_\pi^2/s}$$

$$t_0^0(s) = a + (s - 4M_\pi^2)b + \int_{4M_\pi^2}^{\infty} ds' \{ K_0(s, s') \operatorname{Im} t_0^0(s') \\ + K_1(s, s') \operatorname{Im} t_1^1(s') + K_2(s, s') \operatorname{Im} t_2^2(s') \} \\ + \text{higher partial waves}$$

- The subtraction constants are determined by  $a_0^0, a_0^2$ :

$$a = a_0^0, \quad b = (2a_0^0 - 5a_0^2)/(12M_\pi^2)$$

- The kernels are elementary functions, e.g.

$$K_0(s, s') = \underbrace{\frac{1}{\pi(s' - s)}}_{r.h.cut} + \underbrace{\frac{2 \ln\{(s + s' - 4M_\pi^2)/s'\}}{3\pi(s - 4M_\pi^2)} - \frac{5s' + 2s - 16M_\pi^2}{3\pi s'(s' - 4M_\pi^2)}}_{l.h.cut}$$

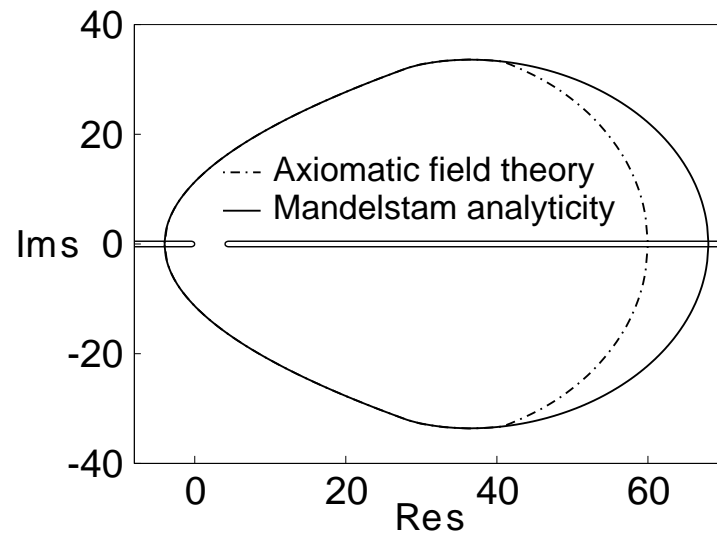
- Left hand cut is essential for convergence:

$$K_0(s, s') \sim 1/s'^3 \text{ for large } s'$$

# Domain of validity of the Roy equations

- Roy derived his equations for real energies in the interval  $-4M_\pi^2 < s < 60M_\pi^2$
- Equations are valid for complex  $s$  in a limited region of the first sheet

I. Caprini, G. Colangelo and H. Leutwyler  
Phys. Rev. Lett. 96 (2006) 132001



$s$  in units of  $M_\pi^2$

- Proof is based on first principles, general quantum field theory

A. Martin, Scattering Theory:  
Unitarity, Analyticity and Crossing  
Lecture Notes in Physics, vol. 3, 1969

G. Mahoux, S. M. Roy and G. Wanders  
Nucl. Phys. B 70 (1974) 297

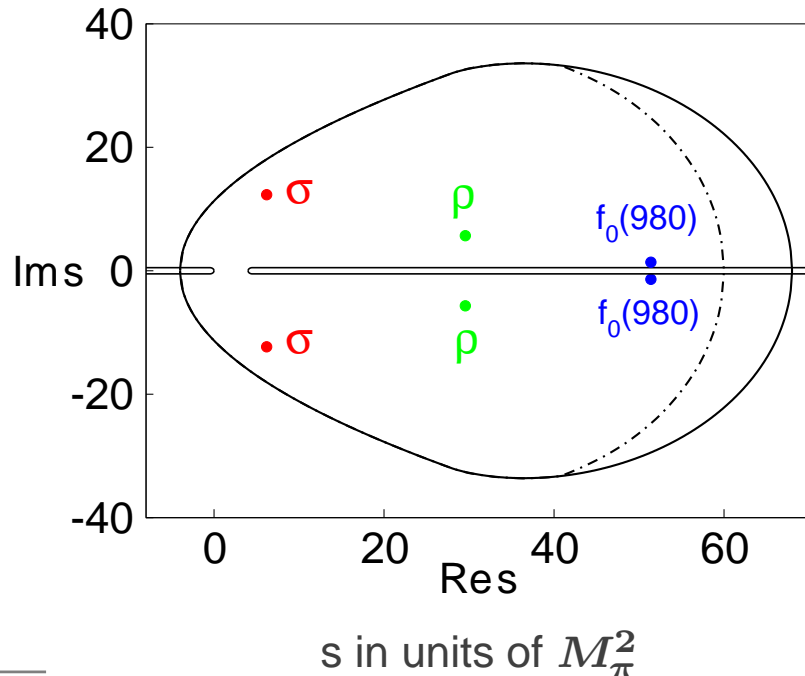
⇒ Exact representation for  $S_0^0(s)$  in this region  
Do not need to parametrize the amplitude

# Evaluation of the pole position

- Insert our solutions of the Roy equations  
For the central solution,  $S_0^0(s)$  has two pairs of zeros in the region of validity of the representation:

$$s = (6.2 \pm i 12.3) M_\pi^2 \quad \sigma$$

$$s = (51.4 \pm i 1.4) M_\pi^2 \quad f_0(980)$$



- ⇒ 1. Lowest resonance of QCD has vacuum quantum numbers
2. Pole on lower half of sheet II occurs in vicinity of

$$m_\sigma = 441 - i 272 \text{ MeV}$$

$$= M_\sigma - \frac{i}{2} \Gamma_\sigma$$

# Error analysis

- Results depend on phenomenological input used when solving the Roy equations, subject to uncertainties  
Can follow error propagation explicitly

- Pole position of  $\sigma$  mainly depends on 3 input variables:

$$a_0^0, a_0^2, \delta_A \equiv \delta_0^0(800 \text{ MeV})$$

- Substantial uncertainties in phenomenology of  $\delta_A$
- Use the range:  $\delta_A = 82.3^\circ \begin{matrix} +10^\circ \\ -4^\circ \end{matrix}$



## Error analysis

- Noise from remaining input variables is very small:

$$m_\sigma^0 = (441 \pm 4) - i(272 \pm 6) \text{ MeV}$$

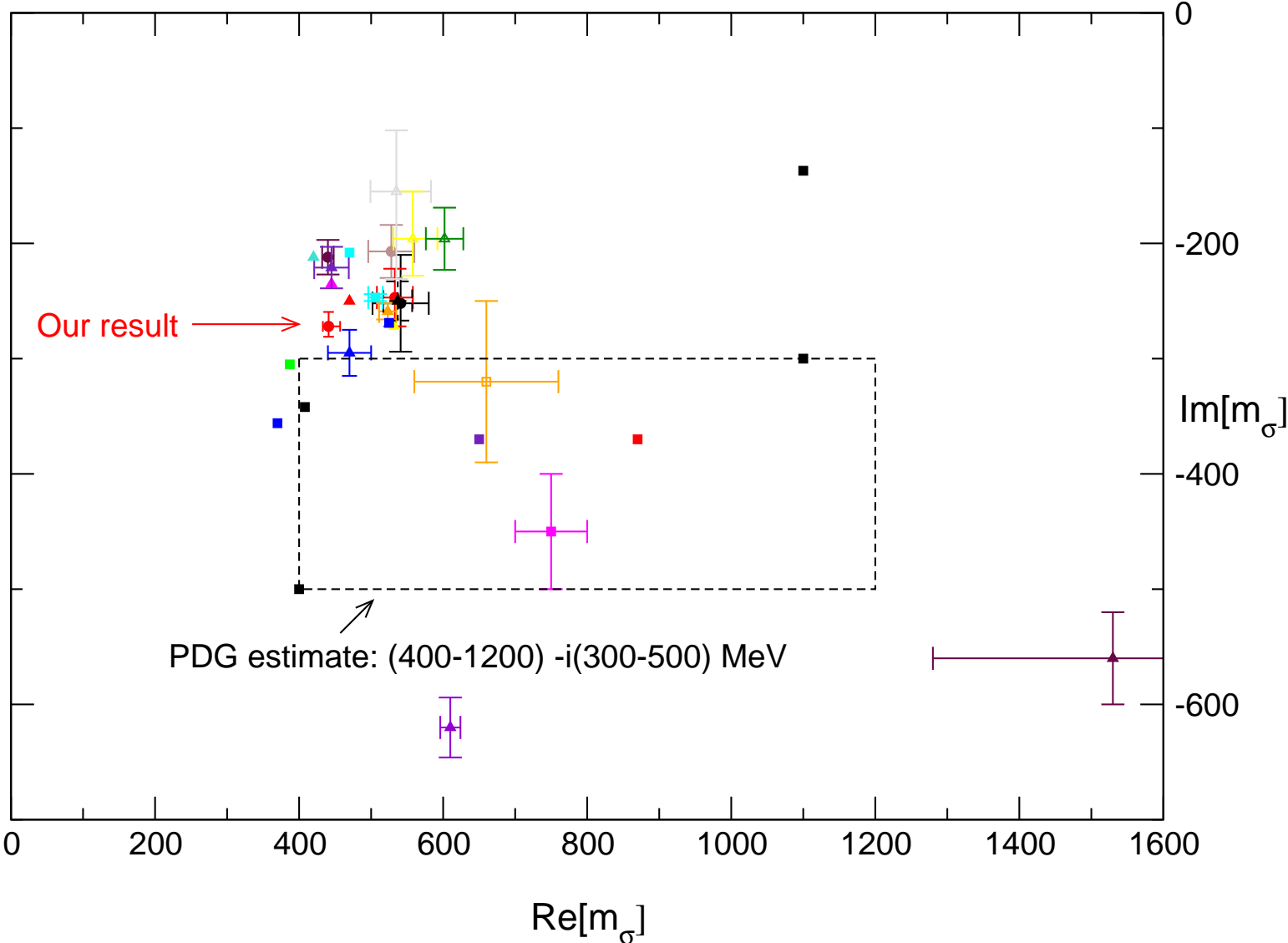
- Values of  $a_0^0$ ,  $a_0^2$ ,  $\delta_A$  are crucial:

$$\begin{aligned} \Delta m_\sigma = & (-2.4 + i 3.8) \frac{a_0^0 - 0.22}{0.005} \\ & + (0.8 - i 4.0) \frac{a_0^2 + 0.0444}{0.001} \\ & + (5.3 + i 3.3) \frac{\delta_A - 82.3}{3.4} \end{aligned} \quad \text{numbers in MeV}$$

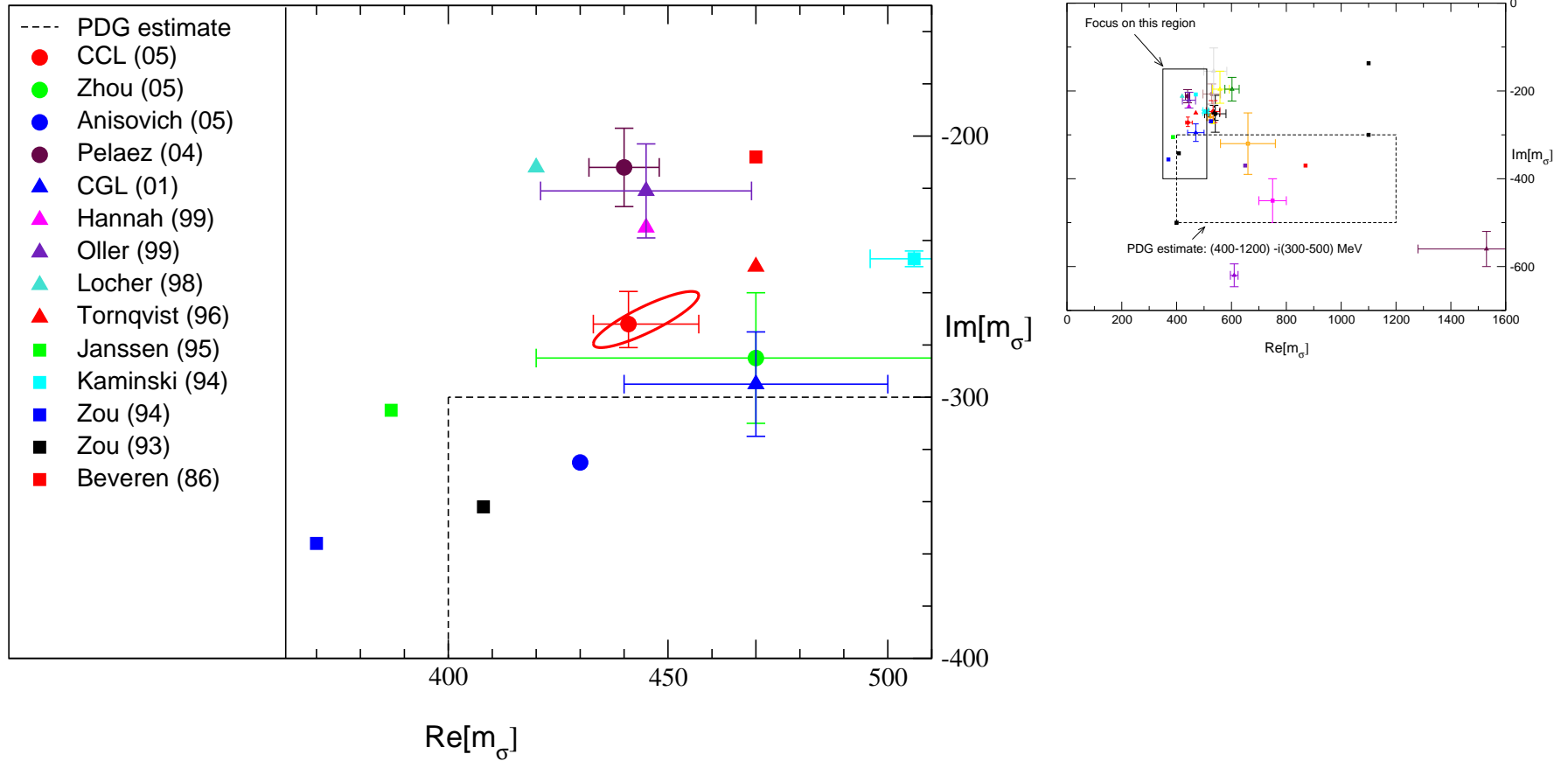
- Final result: insert the predictions for  $a_0^0$ ,  $a_0^2$ , use the phenomenological range for  $\delta_A$  and add errors up:

$$m_\sigma = 441 \begin{matrix} +16 \\ -8 \end{matrix} - i 272 \begin{matrix} +9 \\ -13 \end{matrix} \text{ MeV}$$

# Comparison with compilation of PDG



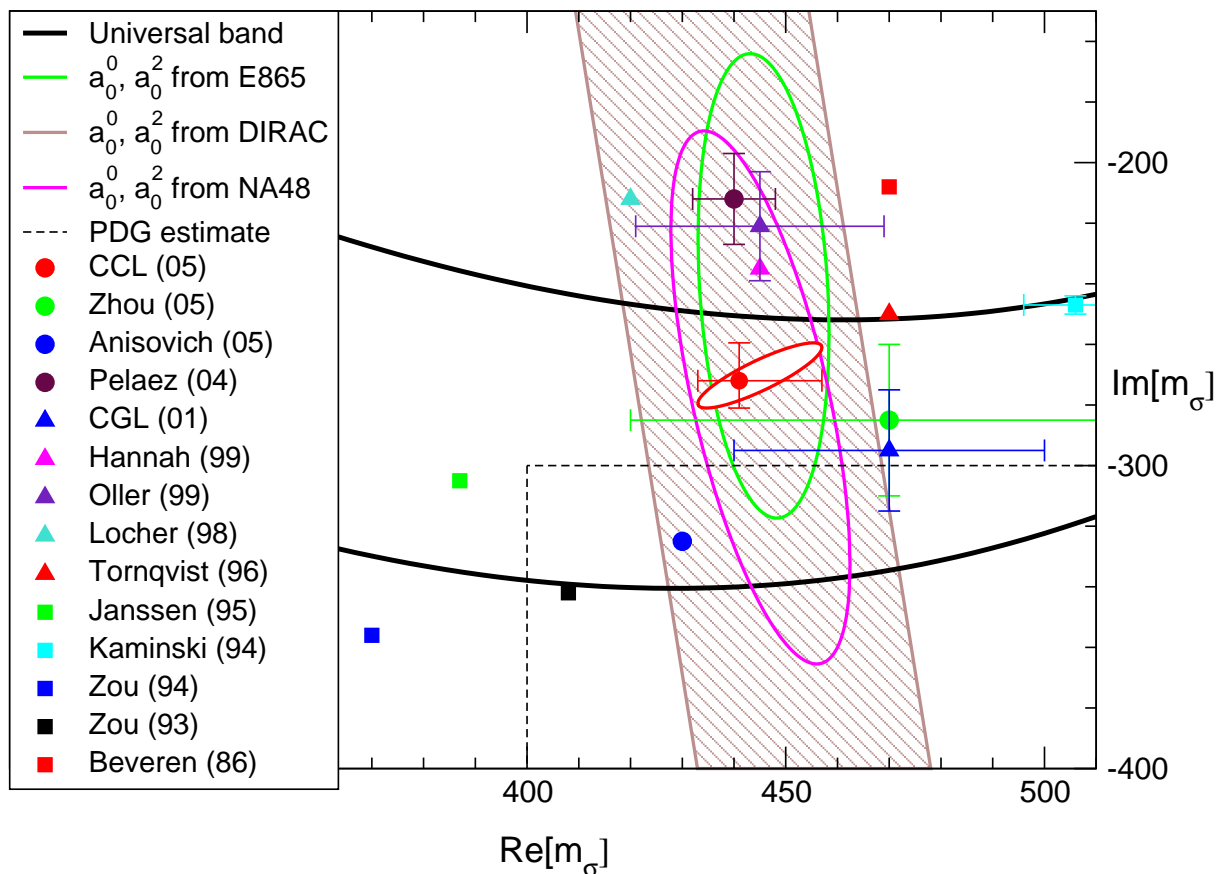
# Vicinity of the pole



Results for  $Re[m_\sigma]$  and  $Im[m_\sigma]$  are strongly correlated

# Ignore the theoretical predictions for $a_0^0, a_0^2$

- Replace the low energy theorems for  $a_0^0, a_0^2$  by the experimental results from E865, DIRAC and NA48
- $a_0^0, a_0^2 \in$  universal band



## Why are our errors so incredibly small ?

- The  $\sigma$  occurs at low energies
- At low energies, the subtraction term dominates

$$t_0^0(s) \simeq a_0^0 + (2a_0^0 - 5a_0^2) \frac{(s - 4M_\pi^2)}{12M_\pi^2}$$

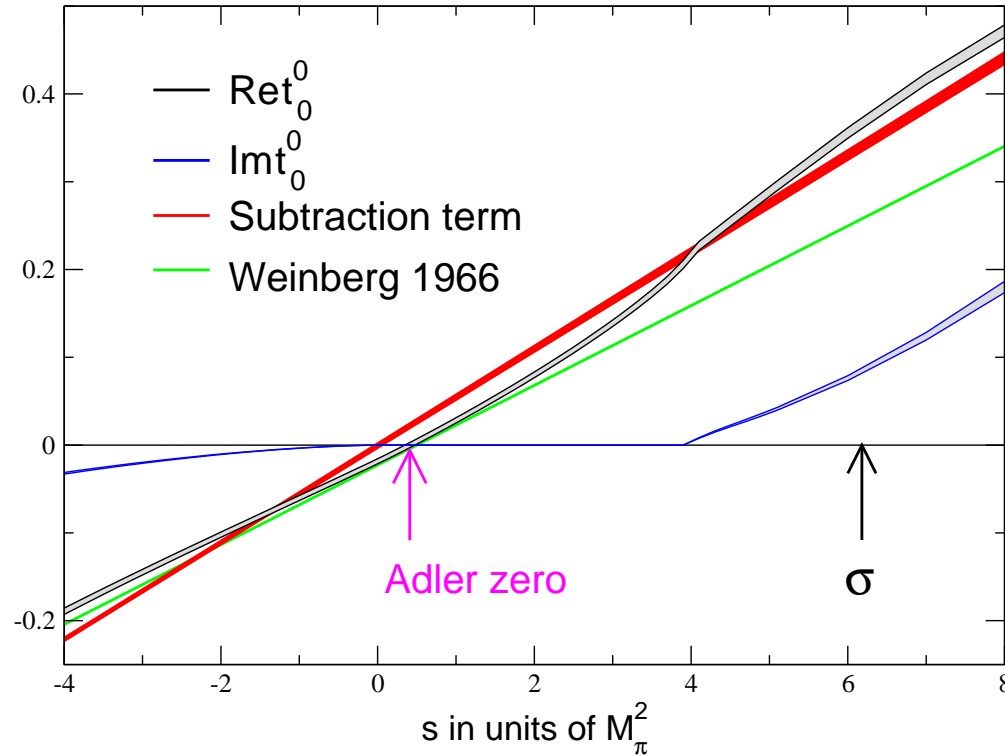
Insert low energy theorem for  $a_0^0, a_0^2$

⇒ Roy equation reduces to Weinberg formula

$$t_0^0(s) \simeq \frac{(2s - M_\pi^2)}{32\pi F_\pi^2}$$

Dispersion integrals only represent a correction

# At low energies, the subtraction term dominates



$$s = (0.41 \pm 0.06) M_\pi^2 \quad \text{Adler zero}$$

$$s = (6.2 - i 12.3) M_\pi^2 \quad \text{pole from } \sigma$$

## Estimate pole position on back of an envelope

- Approximate  $t_0^0(s)$  with the Weinberg formula

$$t_0^0(s) = \frac{(2s - M_\pi^2)}{32\pi F_\pi^2}$$

Where are the zeros of  $S_0^0(s)$  in this approximation ?

$$1 + 2i \sqrt{1 - 4M_\pi^2/s} t_0^0(s) = 0$$

⇒ Cubic equation for  $s$

- Pair of complex zeros,  $m_\sigma = 365 - i 291$  MeV
- Correction from higher orders amounts to

$$\Delta m_\sigma = 76 \begin{matrix} +16 \\ -8 \end{matrix} + i 19 \begin{matrix} +9 \\ -13 \end{matrix} \text{ MeV}$$

For the quantity that counts, the accuracy is modest

- Real zero on sheet II, near  $s = 0$  (full amplitude has kinematic singularity: vanishes on sheet II at  $s = 0$ )

# Conclusion

- Low energy pion physics: theory ahead of experiment
  - Precision experiments carried out and under way
  - Lattice makes slow, but steady progress
  - So far, all tests confirm the theory
- Limitations of our approach:
  - Calculations cannot be done on back of an envelope
  - Method only covers low energies
  - Only a few applications have been worked out:  
 $\pi\pi$  scattering, pion form factors, hadronic vacuum polarization in SM prediction for muon  $g - 2$   
 $\gamma\gamma \rightarrow \pi^0\pi^0$  M. Pennington, hep-ph/0604212
- Much is yet to be done:  $J/\psi \rightarrow \omega\pi\pi$ ,  $D \rightarrow 3\pi$ , ...  
 $\pi K$ ,  $\kappa$ , ...



# Conclusion

- Model independent method for analytic continuation
  - The lowest resonance of QCD occurs at
$$M_\sigma = 441 \begin{matrix} +16 \\ -8 \end{matrix} \text{ MeV} \quad \Gamma_\sigma = 544 \begin{matrix} +18 \\ -25 \end{matrix} \text{ MeV}$$
and carries vacuum quantum numbers
  - Crossing symmetry plays an essential role:  
Fixes contributions from left hand cut  
Ensures fast convergence, low energy dominance
  - Pole occurs at low value of  $s$ , closer to left hand cut than to singularities from  $K\bar{K}$ ,  $f_0(980)$
  - Result for  $\Gamma_\sigma$  relies on theory for  $a_0^2$   
Experiments concerning  $a_0^2$  would be most welcome



# VISIT THE TAMED RED DRAGON

GENTLE ANIMAL  
LOOK IN HIS EYES FROM CLOSE  
SMELL HIS GOOD BREATH  
BRING YOUR PIONS ALONG AND  
FEED HIM YOURSELF

The management denies responsibility for incidents involving the dragon's tail