Symmetries of the strong interaction

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QCD with 3 massless quarks

- As far as the strong interaction goes, the only difference between the quark flavours $u, d, \ldots, t$ is the mass

  $\boxed{m_u, m_d, m_s \text{ happen to be small}}$

- For massless fermions, the right- and left-handed components lead a life of their own

- Fictitious world with $m_u = m_d = m_s = 0$:

  QCD acquires an exact chiral symmetry

  No distinction between u_L, d_L, s_L, nor between u_R, d_R, s_R

$\Rightarrow$ Hamiltonian is invariant under $\mathrm{SU(3)}_L \times \mathrm{SU(3)}_R$

Chiral symmetry of QCD is hidden

- Chiral symmetry is spontaneously broken:

  Ground state is not symmetric under $\mathrm{SU(3)}_L \times \mathrm{SU(3)}_R$ symmetric only under the subgroup $\mathrm{SU(3)}_\mathrm{V} = \mathrm{SU(3)}_{L+R}$

$\Rightarrow$ Mesons and baryons form degenerate $\mathrm{SU(3)}_\mathrm{V}$ multiplets and the lowest multiplet is massless:

$$M_{\pi^\pm} = M_{\pi^0} = M_{K^\pm} = M_{K^0} = M_{\bar{K}^0} = M_\eta = 0$$

Goldstone bosons of the hidden symmetry

Spontane Magnetisierung

- Heisenbergmodell eines Ferromagneten
  - Gitter von Teilchen, Wechselwirkung zwischen Spins
  - Hamiltonoperator ist drehinvariant
  - Im Zustand mit der tiefsten Energie zeigen alle Spins in dieselbe Richtung: Spontane Magnetisierung

  $\Rightarrow$ Grundzustand nicht drehinvariant

  - Symmetrie gegenüber Drehungen gar nicht sichtbar
    Symmetrie ist versteckt, spontan gebrochen
  - Goldstonebosonen in diesem Fall:
    "Magnonen", "Spinwellen"
    Haben keine Energielücke: $\omega \rightarrow 0$ für $\lambda \rightarrow \infty$
- Nambu realisierte, dass auch in der Teilchenphysik Symmetrien spontan zusammenbrechen können

Chiral symmetry

- QCD with three massless quarks is invariant under $G = SU(3)_L \times SU(3)_R$
- SU(3) has 8 parameters

$\Rightarrow$ Symmetry under Lie group with 16 parameters

$\Rightarrow$ 16 conserved “charges”

$Q_1^V, \ldots, Q_8^V$ (vector currents, R+L)

$Q_1^A, \ldots, Q_8^A$ (axial currents, R-L)

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$Q_1^A, \ldots, Q_8^A$ (axial currents, R-L)

- Charges commute with the Hamiltonian:

$$[Q_i^V, H_0] = 0 \qquad [Q_i^A, H_0] = 0$$

- Representation of charges in terms of conserved currents, such as $\bar{u}\gamma^\mu d$ and $\bar{u}\gamma^\mu\gamma_5 d$:

$$Q_1^V = \int d^3x\, \bar{u}\gamma^0 d, \qquad Q_1^A = \int d^3x\, \bar{u}\gamma^0\gamma_5 d$$

Symmetry properties of the ground state

- Vafa and Witten 1984: state of lowest energy is invariant under the vector charges

  $Q_i^{\mathsf{V}}\,|0\rangle = 0$

- Axial charges ? $Q_i^{\mathsf{A}}\,|0\rangle = ?$

Two alternatives for axial charges

$Q_i^{\mathsf{A}}\,|0\rangle = 0$
Wigner-Weyl realization of G ground state is symmetric $\langle 0|\,\bar{q}_{\mathsf{R}}\,q_{\mathsf{L}}\,|0\rangle = 0$ ordinary symmetry spectrum contains parity partners degenerate multiplets of G

$Q_i^{\mathsf{A}}\,|0\rangle \neq 0$
Nambu-Goldstone realization of G ground state is asymmetric $\langle 0|\,\bar{q}_{\mathsf{R}}\,q_{\mathsf{L}}\,|0\rangle \neq 0$ "order parameter" spontaneously broken symmetry spectrum contains Goldstone bosons degenerate multiplets of $\mathrm{SU(3)}_{\mathsf{V}} \subset \mathrm{G}$

$$\mathrm{G} = \mathrm{SU(3)}_{\mathsf{R}} \times \mathrm{SU(3)}_{\mathsf{L}}$$

Goldstone bosons

- QCD chooses the Nambu-Goldstone mode: $Q_i^{\mathsf{A}}\,|0\rangle \neq 0$
- Immediate consequence:

  $H_0\, Q_i^{\mathsf{A}}\,|0\rangle = Q_i^{\mathsf{A}}\, H_0\,|0\rangle = 0 \quad$ for $i = 1, \ldots, 8$

$\Rightarrow$ Spectrum must contain 8 states $Q_1^{\mathsf{A}}\,|0\rangle, \; \ldots \;, \; Q_8^{\mathsf{A}}\,|0\rangle$
with $E = 0$, spin 0, negative parity, octet of SU(3)$_{\mathsf{V}}$
"Goldstone bosons"

Indeed, the 8 lightest mesons do have these quantum numbers:

$\pi^+, \pi^0, \pi^-, K^+, K^0, \bar{K}^0, K^-, \eta$

- All other one-particle states must form degenerate multiplets of SU(3)$_V$

Side remark: mathematics used is slippery

- Argument given for the occurrence of Goldstone bosons is not quite water tight:

$$\langle 0|\, Q_i^{\mathsf{A}}\, Q_k^{\mathsf{A}}\, |0\rangle = \int d^3x d^3y\, \langle 0|\, A_i^0(x)\, A_k^0(y)\, |0\rangle$$

$\langle 0|\, A_i^0(x)\, A_k^0(y)\, |0\rangle$ only depends on $\vec{x} - \vec{y}$

$\Rightarrow$ $\langle 0|\, Q_i^{\mathsf{A}}\, Q_k^{\mathsf{A}}\, |0\rangle$ is proportional to the volume of the universe, $|Q_i^{\mathsf{A}}\, |0\rangle| = \infty$

- Rigorous proof of Goldstone theorem given later

Quark condensate

- $\bar{q}_{\mathsf{R}}\, q_{\mathsf{L}} = \begin{pmatrix} \bar{u}_{\mathsf{R}}\, u_{\mathsf{L}} & \bar{d}_{\mathsf{R}}\, u_{\mathsf{L}} & \bar{s}_{\mathsf{R}}\, u_{\mathsf{L}} \\ \bar{u}_{\mathsf{R}}\, d_{\mathsf{L}} & \bar{d}_{\mathsf{R}}\, d_{\mathsf{L}} & \bar{s}_{\mathsf{R}}\, d_{\mathsf{L}} \\ \bar{u}_{\mathsf{R}}\, s_{\mathsf{L}} & \bar{d}_{\mathsf{R}}\, s_{\mathsf{L}} & \bar{s}_{\mathsf{R}}\, s_{\mathsf{L}} \end{pmatrix}$

  Transforms like $(3, \bar{3})$ under $\mathrm{SU(3)}_{\mathsf{L}} \times \mathrm{SU(3)}_{\mathsf{R}}$

- If the ground state were symmetric, the matrix $\langle 0|\, \bar{q}_{\mathsf{R}}\, q_{\mathsf{L}}\, |0\rangle$ would have to vanish, because it singles out a direction in flavour space

- $\langle 0|\, \bar{q}_{\mathsf{R}}\, q_{\mathsf{L}}\, |0\rangle$ is referred to as the “quark condensate”, quantitative measure of the strength of spontaneous symmetry breaking, “order parameter”

  $\langle 0|\, \bar{q}_{\mathsf{R}}\, q_{\mathsf{L}}\, |0\rangle$ is the analog of magnetization

Quark condensate

- Ground state is invariant under $SU(3)_V$

$\Rightarrow$ $\langle 0|\, \bar{q}_R\, q_L\, |0\rangle$ is proportional to unit matrix

$$\langle 0|\, \bar{u}_R\, u_L\, |0\rangle = \langle 0|\, \bar{d}_R\, d_L\, |0\rangle = \langle 0|\, \bar{s}_R\, s_L\, |0\rangle$$

$$\langle 0|\, \bar{u}_R\, d_L\, |0\rangle = \ldots = 0$$

Quark masses

- Real world $\neq$ paradise
- In reality, the multiplets are split and the lightest mesons are not massless

  $m_u\,,\, m_d\,,\, m_s \neq 0$

  Quark masses break chiral symmetry

  allow the left to talk to the right

$\Rightarrow$ Chiral symmetry broken in two ways:

spontaneously $\langle 0|\,\bar{q}_{\mathsf{R}}\, q_{\mathsf{L}}\,|0\rangle \neq 0$

explicitly $m_u\,,\, m_d\,,\, m_s \neq 0$

Quark masses

- Only the diagonal vector currents are strictly conserved in QCD: $N_u, N_d, N_s, N_c, N_b, N_t \;\longrightarrow$ baryon number, electric charge, strangeness, charm, ...
- It so happens that m_u, m_d, m_s are small

$\Rightarrow$ H_{QCD} has an approximate SU(3)$_L\times$SU(3)$_R$ symmetry

- Masses of the light quarks enter the Hamiltonian via

$$H_{\text{QCD}} = H_0 + H_1$$

$$H_1 = \int d^3x\,\{m_u\,\bar{u}u + m_d\,\bar{d}d + m_s\,\bar{s}s\}$$

H_0 describes u, d, s as massless, c, b, t as massive

H_0 is invariant under SU(3)$_L\times$SU(3)$_R$

Pattern of light quark masses

- With the discovery of QCD, the mass of the quarks became an unambiguous concept: quark masses occur in the Hamiltonian of the theory.
- First crude estimate within QCD relied on a model for the wave functions of π, K, ρ, based on SU(6) (spin-flavour-symmetry)

$$\frac{1}{2}(m_u+m_d)=\frac{F_\pi M_\pi^2}{3F_\rho M_\rho}\simeq 5\,\text{MeV}, \qquad m_s\simeq 135\,\text{MeV}$$

"Is the quark mass as small as 5 MeV ?" 1974

- Not very different from the pattern found within the Nambu-Jona-Lasinio model (1961) or the one obtained from sum rules by Okubo (1969)

Notion of quark mass

- Quarks do not occur in isolation
- Mass not directly observable like mass of electron

$\Rightarrow$ Quark mass is a theoretical notion

Bare masses in the Hamiltonian cannot serve to define m_u, m_d, m_s, need to be renormalized

- Quark masses depend on renormalization convention
- Generally accepted convention: compare values in $\overline{\text{MS}}$ scheme (based on dimensional regularization)

  In this scheme, the quark masses depend on the running scale μ, much like the coupling constant g_s
- Estimates quoted in the 2008 edition of the PDG tables:

  $\frac{1}{2}(m_u + m_d) = 2.5 \div 5$ MeV, $\qquad m_s = 70 \div 130$ MeV

  refer to $\mu = 2$ GeV

Pattern of light quark masses

- Difference between m_u and m_d ?
  - e.m. self energy: proton $>$ neutron

  $\Rightarrow$ $M_n > M_p$ cannot be due to the e.m. interaction
  - $M_n > M_p$ must be due to $m_d > m_u$

  $\Rightarrow$ Isospin is not a symmetry of the strong interaction !

  Gasser & L. 1975

  In fact a very strong breaking appears to be needed:

  $m_u \simeq 2.5$ MeV, $m_d \simeq 5$ MeV PDG 2008

Crude picture for m_u, m_d, m_s

- $m_u \simeq 2.5\,\text{MeV}, \;\; m_d \simeq 5\,\text{MeV}, \;\; m_s \simeq 100\,\text{MeV}$
  - m_u and m_d are very different
  - m_u and m_d are small compared to m_s
  - “constituent masses” $\notin$ Lagrangian of QCD

Approximate symmetries are natural in QCD

- Why is isospin such a good quantum number ?

  (a) Dimensional transmutation, logarithmic divergences of perturbation theory $\in$ physics

$\Rightarrow$ QCD has an intrinsic scale

  (b) $m_d - m_u \ll$ scale of QCD, not $\ll m_u + m_d$

- Why is the eightfold way a decent approximate symmetry ?

  $m_s - m_u \ll$ scale of QCD

- Isospin is an even better symmetry because

  $m_d - m_u \ll m_s - m_u$

- $m_u \ll m_s \Rightarrow m_u, m_d, m_s \ll$ scale of QCD

$\Rightarrow$ Masses of the light quarks represent perturbations

  Can neglect these in a first approximation

Quark masses as perturbations

$$H_{\text{QCD}} = H_0 + H_1$$

$$H_1 = \int d^3x \, \{m_u \, \bar{u}u + m_d \, \bar{d}d + m_s \, \bar{s}s\}$$

Expansion in powers of m_u, m_d, m_s	$\Longleftrightarrow$	Perturbation series in powers of H_1

- H_0 treats π, K, η as massless, H_1 gives them a mass

Magnitude of the perturbations due to m_u, m_d, m_s

- $\langle 0|\,\bar{d}\gamma^\mu\gamma_5 u|\pi^+\rangle = i\,p^\mu\sqrt{2}\,F_\pi$

  $\langle 0|\,\bar{s}\gamma^\mu\gamma_5 u|K^+\rangle = i\,p^\mu\sqrt{2}\,F_K$

  Value of F_π, F_K known from $\pi^+ \rightarrow \mu^+\nu$, $K^+ \rightarrow \mu^+\nu$ (and CKM matrix elements V_{ud}, V_{us})
- Difference between π^+ and K^+ comes from $m_s > m_d$
- Observed ratio: $\dfrac{F_K}{F_\pi} = 1.19 \pm 0.01$

  Branching fraction of $K \rightarrow \pi e \nu$ changed by $> 3\,\sigma$ in 2004 ! $1.22 \rightarrow 1.19$

$\Rightarrow$ $m_s - m_d$ generates correction of order 20%

- $m_u, m_d \ll m_s$ $\Rightarrow$ correction mainly comes from m_s
- effects from m_u, m_d are tiny

Gell-Mann-Oakes-Renner formula

- First order perturbation theory yields:

$$M_\pi^2 = (m_u + m_d) \times |\langle 0|\, \bar{u}u \,|0\rangle| \times \frac{1}{F_\pi^2}$$

$$\Uparrow \qquad\qquad\qquad \Uparrow$$

explicit spontaneous

Gell-Mann, Oakes & Renner 1968. At that time, the quarks very still considered with suspicion $\Rightarrow$ formula does not appear like this in the paper

- Postpone derivation (involves Goldstone theorem)

Consequences of GMOR formula

- $M_\pi^2 = (m_u + m_d)\,B + O(m^2)$

$\Rightarrow$ The energy gap of QCD is small because m_u, m_d happen to be small

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- Goldstone boson masses measure the strength of chiral symmetry breaking

- π and K belong to an octet of SU(3)$_{\text{V}}$

$\Rightarrow$ Goldstone boson masses strongly violate SU(3)$_{\text{V}}$

Check of SU(3)$_V$

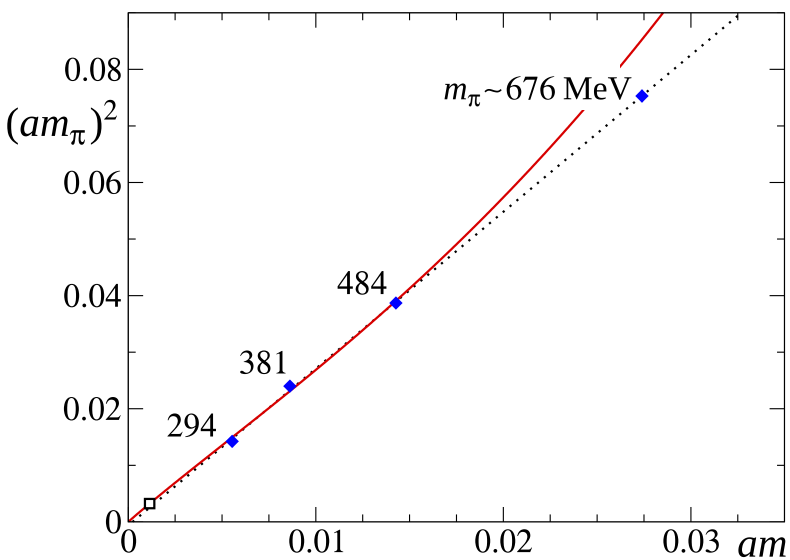
- Goldstone boson masses strongly break SU(3)$_V$
  Nevertheless, SU(3)$_V$ is a decent approximate symmetry
- Check: first order perturbation theory also yields
  $$M_\eta^2 = \frac{1}{3}(m_u + m_d + 4m_s)B + O(m^2)$$

$$\Rightarrow M_\pi^2 - 4M_K^2 + 3M_\eta^2 = O(m^2)$$

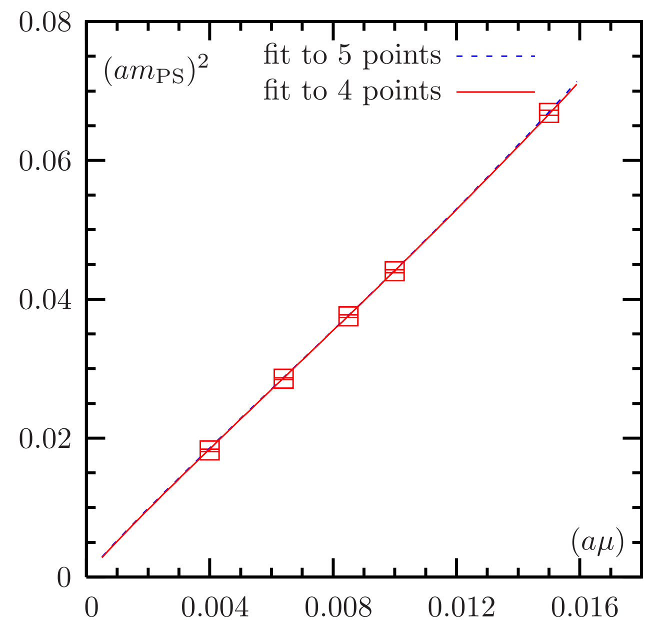
Gell-Mann-Okubo formula for M^2 $\surd$

Lattice

- Simulations of QCD on a lattice now reach sufficiently small lattice spacings, sufficiently small quark masses to make contact with physics
- GMOR formula can now be checked on the lattice: determine M_π as a function of $m_u = m_d = m$



Lüscher, Lattice conference 2005



ETM collaboration, hep-lat/0701012

Lattice

- Quality of data is impressive
- No quenching, quark masses are sufficiently light

$\Rightarrow$ Legitimate to use χPT for the extrapolation to the physical values of m_u, m_d

- Proportionality of M_π^2 to the quark mass appears to hold out to values of m_u, m_d that are an order of magnitude larger than in nature
- Main limitation: systematic uncertainties
  in particular: $N_f = 2 \rightarrow N_f = 3$

Summary

H_{QCD} has an approximate symmetry: $G = \text{SU}(3)_L \times \text{SU}(3)_R$

$|0\rangle$ approximately symmetric only under $\text{SU}(3)_V \subset G$

- World we live in is close to the paradise
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$\Rightarrow$ Only the subgroup $\text{SU(3)}_V \subset G$ is an approximate symmetry of spectrum and matrix elements

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"Eightfold way", $u \leftrightarrow d \leftrightarrow s$

- m_u, m_d are particularly small

$\Rightarrow$ $\text{SU}(2)_L \times \text{SU}(2)_R$ is a nearly exact symmetry of H_{QCD}

$\Rightarrow$ Expansion in powers of m_u, m_d converges very rapidly

Appendix: Proof of Goldstone theorem

$\langle 0|\, \bar{q}_{\mathsf{R}}\, q_{\mathsf{L}}\, |0\rangle \neq 0 \Rightarrow \exists$ massless particles

- $$Q = \int d^3x\, \bar{u}\gamma^0\gamma_5 d$$

  $$[Q, \bar{d}\gamma_5 u] = -\bar{u}u - \bar{d}d$$

- First determine the form of the 2-point-function

  $$F^\mu(x-y) \equiv \langle 0|\, \bar{u}(x)\gamma^\mu\gamma_5 d(x)\, \bar{d}(y)\gamma_5 u(y)\, |0\rangle$$

  Lorentz invariance $\Rightarrow F^\mu(z) = z^\mu f(z^2)$

  Chiral symmetry $\Rightarrow \partial_\mu F^\mu(z) = 0$

$\Rightarrow F^\mu(z) = \dfrac{z^\mu}{z^4} \times$ constant (for $z^2 \neq 0$)

$\Rightarrow$ Symmetry fixes the 2-point-function up to a constant

Proof of Goldstone theorem ctd.

- Compare Källen–Lehmann representation:

$$\langle 0|\,\bar{u}(x)\gamma^\mu\gamma_5 d(x)\,\bar{d}(y)\gamma_5 u(y)\,|0\rangle$$

$$= (2\pi)^{-3}\!\int\! d^4p\, p^\mu\,\rho(p^2)e^{-ip(x-y)}$$

$$= \int_0^\infty\! ds\,\rho(s)\partial^\mu\Delta^+(x-y,s)$$

- $\Delta^+(z,s)$: positive frequency part of propagator

$$\Delta^+(z,s) = \frac{i}{(2\pi)^3}\int\! d^4p\,\theta(p^0)\,\delta(p^2-s)\,e^{-ipz}$$

- Massless propagator:

$$\Delta^+(z,0) = \frac{1}{4\pi\, i\, z^2} \qquad \Rightarrow \partial_\mu\Delta^+(z,0) = \frac{z_\mu}{z^4}\times\text{constant}$$

Proof of Goldstone theorem ctd.

- Result:

$$\langle 0|\,\bar{u}(x)\gamma^\mu\gamma_5 d(x)\,\bar{d}(y)\gamma_5 u(y)\,|0\rangle = C\,\partial^\mu\Delta^+(z,0)$$

$\Rightarrow$ Only massless intermediate states contribute:

$$\rho(s) = C\,\delta(s)$$

- Why only massless intermediate states ?
  - $\langle 0|\,\bar{u}(x)\gamma^\mu\gamma_5 d(x)\,\bar{d}(y)\gamma_5 u(y)\,|0\rangle$
    $= \sum_n \langle 0|\,\bar{u}\gamma^\mu\gamma_5 d|n\rangle\langle n|\,\bar{d}\gamma_5 u\,|0\rangle\; e^{-i\,p_n(x-y)}$
  - $\langle n|\,\bar{d}\gamma_5 u\,|0\rangle \neq 0$ only if $\langle n|$ has spin 0
  - If $|n\rangle$ has spin 0 $\Rightarrow \langle 0|\,\bar{u}(x)\gamma^\mu\gamma_5 d(x)|n\rangle \propto p^\mu\, e^{-ipx}$
  - $\partial_\mu(\bar{u}\gamma^\mu\gamma_5 d) = 0 \Rightarrow p^2 = 0$

$\Rightarrow$ Either $\exists$ massless particles or $C = 0$

Proof of Goldstone theorem ctd.

- Claim: $\langle 0|\,\bar{q}_\mathsf{R}\, q_\mathsf{L}|0\rangle \neq 0 \Rightarrow C \neq 0$
  - Interchange the two operators:

    $\langle 0|\,\bar{d}(y)\gamma_5 u(y)\,\bar{u}(x)\gamma^\mu\gamma_5 d(x)\,|0\rangle = C'\,\partial^\mu\Delta^-(z)$

    $\Rightarrow \langle 0|\,[\bar{u}(x)\gamma^\mu\gamma_5 d(x),\, \bar{d}(y)\gamma_5 u(y)]\,|0\rangle$

    $= C\partial^\mu\Delta^+(z,0) - C'\partial^\mu\Delta^-(z,0)$
  - Causality: if $x-y$ is spacelike, then

    $\langle 0|\,[\bar{u}(x)\gamma^\mu\gamma_5 d(x),\, \bar{d}(y)\gamma_5 u(y)]\,|0\rangle = 0$

    $\Rightarrow C' = -C$

    $\Rightarrow \langle 0|\,[\bar{u}(x)\gamma^\mu\gamma_5 d(x),\, \bar{d}(y)\gamma_5 u(y)]\,|0\rangle = C\partial^\mu\Delta(z,0)$

    $\Rightarrow \langle 0|\,[\,Q,\, \bar{d}(y)\gamma_5 u(y)]\,|0\rangle = C$
  - $\langle 0|\,[\,Q,\, \bar{d}(y)\gamma_5 u(y)]\,|0\rangle = -\langle 0|\bar{u}u + \bar{d}d\,|0\rangle$
- Hence $\langle 0|\bar{u}u + \bar{d}d\,|0\rangle \neq 0$ implies $C \neq 0$ qed.

Appendix: Derivation of GMOR relation

- Pion matrix elements in massless theory:

$$\langle 0|\,\bar{u}\gamma^\mu\gamma_5 d|\pi^-\rangle = i\,\sqrt{2}\,F\,p^\mu$$

$$\langle 0|\,\bar{u}\,i\,\gamma_5 d|\pi^-\rangle = \sqrt{2}\,G$$

Only the one–pion intermediate state

$$\langle 0|\,\bar{u}(x)\gamma^\mu\gamma_5 d(x)\,\bar{d}(y)\gamma_5 u(y)\,|0\rangle = C\,\partial^\mu\Delta^+(z,0)$$

$$\Uparrow$$

$$|\pi^-\rangle\langle\pi^-|$$

contributes. Hence $2\,F\,G = C$

- Value of C fixed by quark condensate

$$C = -\langle 0|\,\bar{u}u + \bar{d}d\,|0\rangle$$

- Exact result in massless theory:

$$\boxed{F\,G = -\langle 0|\,\bar{u}u\,|0\rangle}$$

Derivation of GMOR relation ctd.

- Turn quark masses on

$$\langle 0|\bar{u}\gamma^\mu\gamma_5 d|\pi^-\rangle = i\sqrt{2}\,F_\pi\,p^\mu$$

$$\langle 0|\bar{u}\,i\,\gamma_5 d|\pi^-\rangle = \sqrt{2}\,G_\pi$$

- Current conservation

$$\partial_\mu(\bar{u}\gamma^\mu\gamma_5 d) = (m_u + m_d)\,\bar{u}\,i\,\gamma_5 d$$

$$\Rightarrow F_\pi M_\pi^2 = (m_u + m_d)\,G_\pi$$

$$\boxed{M_\pi^2 = (m_u + m_d)\,\frac{G_\pi}{F_\pi}}$$ exact for $m \neq 0$

Proof of GMOR relation ctd.

- Collect the results
  - $F_\pi \longrightarrow F\,, \quad G_\pi \longrightarrow G$ for $m \longrightarrow 0$
  - $F\,G = -\langle 0|\,\bar{u}u\,|0\rangle$

$$\Rightarrow \frac{G_\pi}{F_\pi} = -\frac{\langle 0|\,\bar{u}u\,|0\rangle}{F_\pi^2} + O(m)$$

- $M_\pi^2 = (m_u + m_d)\,\dfrac{G_\pi}{F_\pi}$

$$\Rightarrow M_\pi^2 = (m_u + m_d)\left(\frac{-\langle 0|\,\bar{u}u\,|0\rangle}{F_\pi^2}\right) + O(m^2)\ \checkmark$$