$\pi\pi$ scattering

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Workshop on Light Flavors and Chiral Dynamics

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$\pi\pi$ interaction

- Main experiments on $\pi\pi$ scattering were done in the seventies. What's new ?
  - New precision data:

$K \to \pi\pi e\nu$	E865	Brookhaven
	NA48/2	CERN
pionic atoms	DIRAC	CERN
$K \to 3\pi$	NA48/2	CERN

  - Lattice results on M_π, F_π, a_0^2, $\langle r^2 \rangle_s$
  - Significant theoretical progress, based on χPT + dispersion theory

Theory of $\pi\pi$ interaction

- $\pi\pi$ scattering is special: crossed channels are identical

$\Rightarrow$ Re $T(s,t)$ can be represented as a twice subtracted dispersion integral over Im $T(s,t)$ in physical region

S.M. Roy 1971

- The 2 subtraction constants can be identified with the S-wave scattering lengths:

$$a_0^0 \ , \ a_0^2 \begin{array}{l} \leftarrow \text{isospin} \\ \leftarrow \text{angular momentum} \end{array}$$

- Representation leads to dispersion relations for the individual partial waves: *Roy equations*

Roy equations

- Pioneering work on the physics of the Roy equations was done around the time when QCD was discovered

  Pennington & Protopopescu 1973, Basdevant, Froggatt & Petersen 1974

- Dispersion integrals converge rapidly (2 subtractions)

$\Rightarrow$ Crude phenomenological information on Im $T(s,t)$ for energies above 800 MeV suffices

$\Rightarrow$ Given a_0^0, a_0^2, the scattering amplitude can be calculated very accurately

Ananthanarayan, Colangelo, Gasser & L. 2001
Descotes, Fuchs, Girlanda & Stern 2002

$\Rightarrow$ a_0^0, a_0^2 are the essential parameters at low energy

- Main problem in early work: a_0^0, a_0^2 poorly known
  Experimental information near threshold is meagre

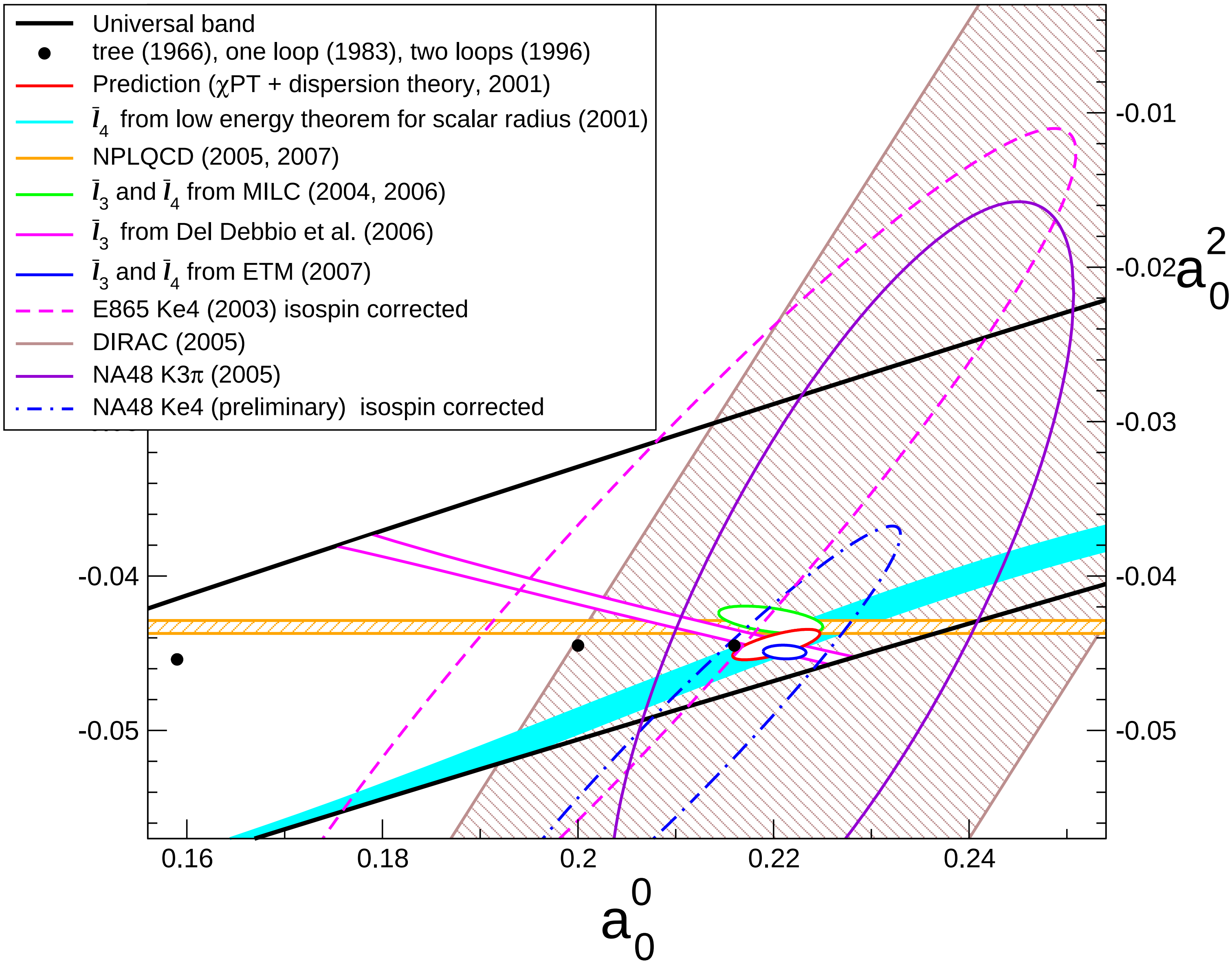
Low energy theorems

- Chiral perturbation theory provides the missing piece: sharp theoretical prediction for a_0^0, a_0^2

  see slides of the talk given at ICFP 2007, www.leutwyler.itp.unibe.ch

- In combination with the low energy theorems for a_0^0, a_0^2, the dispersion relations for the partial waves fix the $\pi\pi$ scattering amplitude to an incredible degree of accuracy

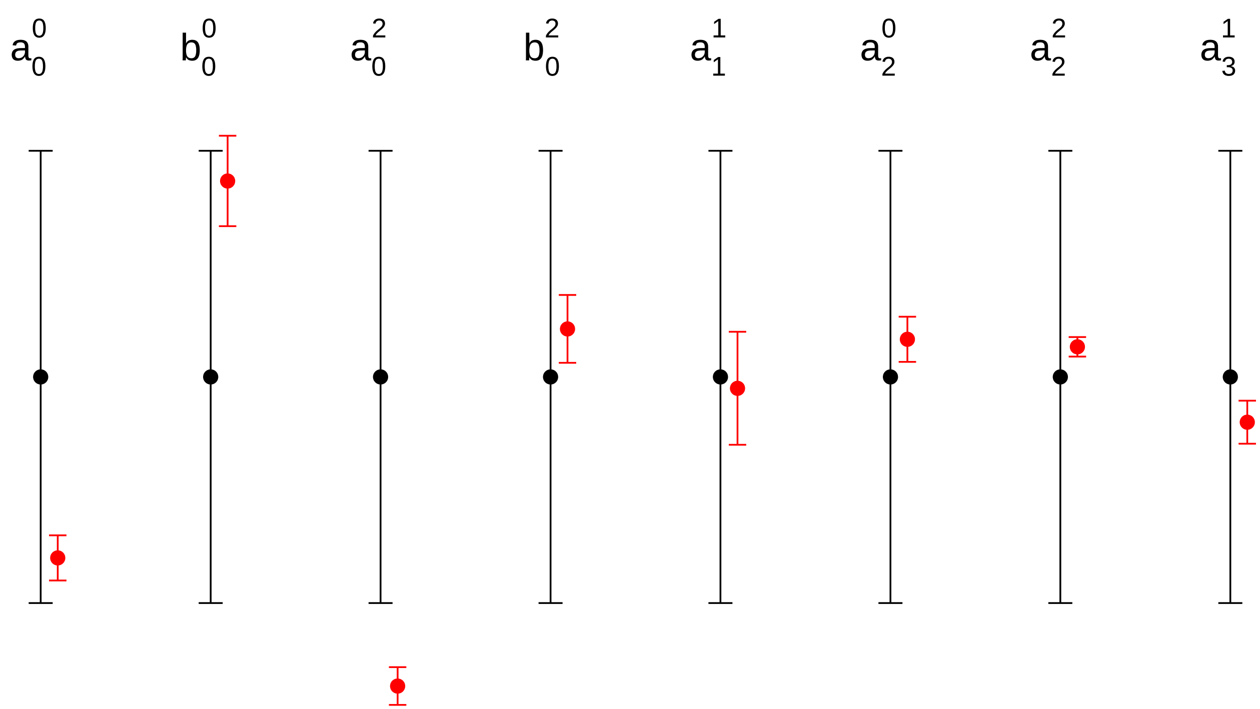
a_0^0, a_0^2 : prediction, lattice & experiment



Theory is ahead of experiment . . .

What difference does is make whether or not the subtraction constants are known accurately ?

- Results obtained on the basis of the experimental information about a_0^0 and a_0^2 Nagels et al. 1979
- Results obtained with the chiral predictions for a_0^0 and a_0^2 Colangelo, Gasser & L. 2001



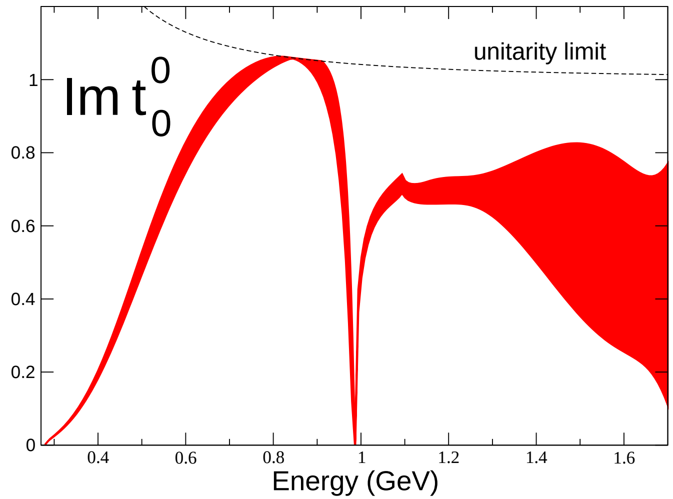
Where is the lowest resonance of QCD ?

I. Caprini, G. Colangelo and H. Leutwyler, Phys. Rev. Lett. 96 (2006) 132001

- Does QCD have a resonance near threshold ?
- Why care ?
  - Concerns the nonperturbative domain of QCD
  - Quark and gluon degrees of freedom useless there

  $\Rightarrow$ Understanding very poor, pattern of energy levels ?
  - Lowest resonance: σ ? ρ ?
- Resonance $\leftrightarrow$ pole on second sheet
  - Poles are universal
  - Pole position is unambiguous, even if width is large
  - Where is the pole closest to the origin ?

The red dragon



There is the broad object seen in $\pi\pi$ scattering, often called "background", which extends from about 400 MeV up to about 1700 MeV. This object we consider as a single broad resonance[2] *which we identify as the lightest glueball with quantum numbers* $J^{PC} = 0^{++} \ldots$

[2] *we refer to it as red dragon*

P. Minkowski and W. Ochs, Eur. Phys. J. C9 (1999) 283

$f_0(600)$ T-MATRIX POLE $\sqrt{s}$

Note that $\Gamma \approx 2\ \mathrm{Im}(\sqrt{s_{\mathrm{pole}}})$.

VALUE (MeV)	*DOCUMENT ID*		*TECN*	*COMMENT*
(400–1200)−*i*(250–500) OUR ESTIMATE				
• • • We do not use the following data for averages, fits, limits, etc. • • •				
$(441^{+16}_{-8})-i(272^{+9}_{-12.5})$	[1] CAPRINI	06	RVUE	$\pi\pi \to \pi\pi$
$(470 \pm 50)-i(285 \pm 25)$	[2] ZHOU	05	RVUE	
$(541 \pm 39)-i(252 \pm 42)$	[3] ABLIKIM	04A	BES2	$J/\psi \to \omega\pi^+\pi^-$
$(528 \pm 32)-i(207 \pm 23)$	[4] GALLEGOS	04	RVUE	Compilation
$(440 \pm 8)-i(212 \pm 15)$	[5] PELAEZ	04A	RVUE	$\pi\pi \to \pi\pi$
$(533 \pm 25)-i(247 \pm 25)$	[6] BUGG	03	RVUE	
$532 - i272$	BLACK	01	RVUE	$\pi^0\pi^0 \to \pi^0\pi^0$
$(470 \pm 30)-i(295 \pm 20)$	[1] COLANGELO	01	RVUE	$\pi\pi \to \pi\pi$
$(535^{+48}_{-36})-i(155^{+76}_{-53})$	[7] ISHIDA	01		$\Upsilon(3S) \to \Upsilon\pi\pi$
$610 \pm 14 - i620 \pm 26$	[8] SUROVTSEV	01	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$(558^{+34}_{-27})-i(196^{+32}_{-41})$	ISHIDA	00B		$p\overline{p} \to \pi^0\pi^0\pi^0$
$445 - i235$	HANNAH	99	RVUE	π scalar form factor
$(523 \pm 12)-i(259 \pm 7)$	KAMINSKI	99	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$, $\sigma\sigma$
$442 - i\ 227$	OLLER	99	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$469 - i203$	OLLER	99B	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$445 - i221$	OLLER	99C	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$, $\eta\eta$
$(1530^{+90}_{-250})-i(560 \pm 40)$	ANISOVICH	98B	RVUE	Compilation
$420 - i\ 212$	LOCHER	98	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$(602 \pm 26)-i(196 \pm 27)$	[9] ISHIDA	97		$\pi\pi \to \pi\pi$
$(537 \pm 20)-i(250 \pm 17)$	[10] KAMINSKI	97B	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$, 4π
$470 - i250$	[11,12] TORNQVIST	96	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$, $K\pi$, $\eta\pi$
$\sim (1100 - i300)$	AMSLER	95B	CBAR	$\overline{p}p \to 3\pi^0$
$400 - i500$	[12,13] AMSLER	95D	CBAR	$\overline{p}p \to 3\pi^0$
$1100 - i137$	[12,14] AMSLER	95D	CBAR	$\overline{p}p \to 3\pi^0$
$387 - i305$	[12,15] JANSSEN	95	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$525 - i269$	[16] ACHASOV	94	RVUE	$\pi\pi \to \pi\pi$
$(506 \pm 10)-i(247 \pm 3)$	KAMINSKI	94	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$370 - i356$	[17] ZOU	94B	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$408 - i342$	[12,17] ZOU	93	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$870 - i370$	[12,18] AU	87	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$470 - i208$	[19] BEVEREN	86	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$, $\eta\eta$, ...
$(750 \pm 50)-i(450 \pm 50)$	[20] ESTABROOKS	79	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$(660 \pm 100)-i(320 \pm 70)$	PROTOPOP...	73	HBC	$\pi\pi \to \pi\pi$, $K\overline{K}$
$650 - i370$	[21] BASDEVANT	72	RVUE	$\pi\pi \to \pi\pi$

Model independent determination of the pole

- Most of the results quoted by the PDG are obtained by
  (a) parametrizing the data for real values of s
  (b) continuing this parametrization analytically in s

  $\Rightarrow$ Result is sensitive to the parametrization used

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- We found a model independent method:
  1. Poles on second sheet are zeros on first sheet
  2. The Roy equations are valid for complex values of s, in a limited region of the first sheet

  $\Rightarrow$ Exact representation of the partial waves in terms of observable quantities, valid for complex values of s

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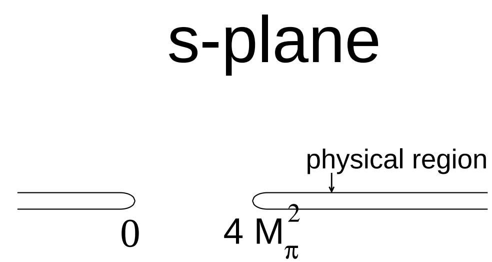
  3. Can evaluate this representation to good precision and determine the zeros numerically

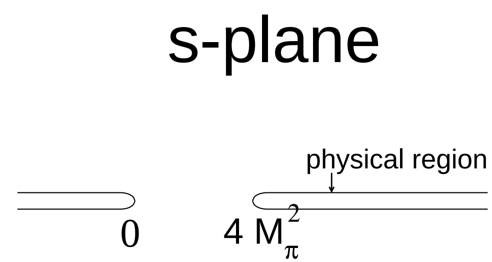
Pole on second sheet $\longleftrightarrow$ zero on first sheet

- $S_0^0(s) = \eta_0^0(s)\exp 2i\delta_0^0(s)$

  $S_0^0(s)$ is analytic in the cut plane



Pole on second sheet ↔ zero on first sheet



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$\Rightarrow$ $S_0^0(s^\star) = S_0^0(s)^\star$

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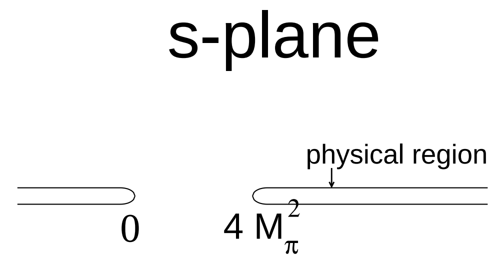

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- Second sheet is reached by continuation across the elastic interval of the right hand cut

$$S_0^0(x - i\epsilon)^{II} = S_0^0(x + i\epsilon)^{I} = 1/S_0^0(x - i\epsilon)^{I}$$

Analyticity $\Rightarrow$ $\boxed{S_0^0(s)^{II} = 1/S_0^0(s)^{I}}$ valid $\forall s$

Pole in $S_0^0(s)^{II}$ $\Longleftrightarrow$ zero in $S_0^0(s)^{I}$

Roy equation for the isoscalar S-wave

$$S_0^0(s) = 1 + 2\,i\rho\,t_0^0(s) \qquad\qquad \rho = \sqrt{1 - 4M_\pi^2/s}$$

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$$t_0^0(s) = a + (s - 4M_\pi^2)\, b + \int_{4M_\pi^2}^{\infty} ds' \left\{ K_0(s,s')\, \mathrm{Im}\, t_0^0(s') \right.$$

$$\left. + K_1(s,s')\, \mathrm{Im}\, t_1^1(s') + K_2(s,s')\, \mathrm{Im}\, t_0^2(s') \right\}$$

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- The subtraction constants are determined by a_0^0, a_0^2:
  $a = a_0^0\,, \qquad b = (2a_0^0 - 5a_0^2)/(12M_\pi^2)$

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$$+K_1(s,s')\,\mathrm{Im}\,t_1^1(s')+\;K_2(s,s')\,\mathrm{Im}\,t_0^2(s')\}$$

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- The kernels are elementary functions, e.g.

$$K_0(s,s')=\underbrace{\frac{1}{\pi(s'-s)}}_{r.h.cut}+\underbrace{\frac{2\,ln\{(s+s'-4M_\pi^2)/s'\}}{3\pi(s-4M_\pi^2)}-\frac{5s'+2s-16M_\pi^2}{3\pi s'(s'-4M_\pi^2)}}_{l.h.cut}$$

Roy equation for the isoscalar S-wave

$$S^0_0(s) = 1 + 2\,i\rho\, t^0_0(s) \qquad\qquad \rho = \sqrt{1 - 4M_\pi^2/s}$$

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- Left hand cut is essential for convergence:
  $K_0(s,s') \sim 1/{s'}^3$ for large s'

Domain of validity of the Roy equations

- Roy derived his equations for real energies in the interval $-4M_\pi^2 < s < 60M_\pi^2$

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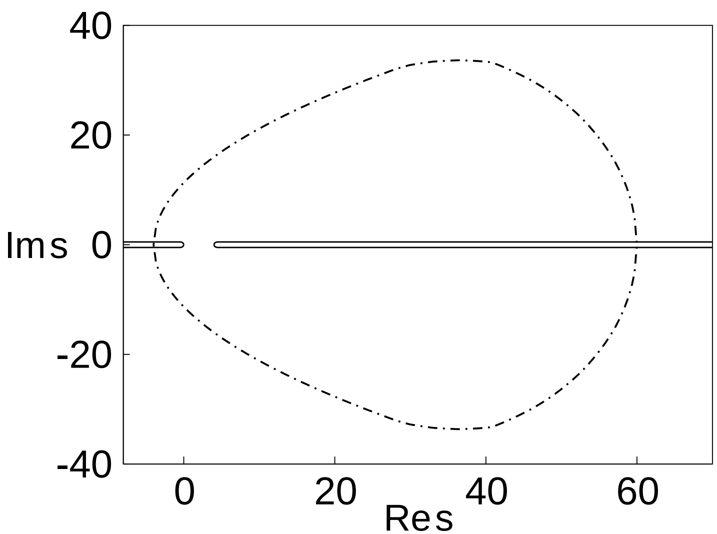

s in units of M_π^2

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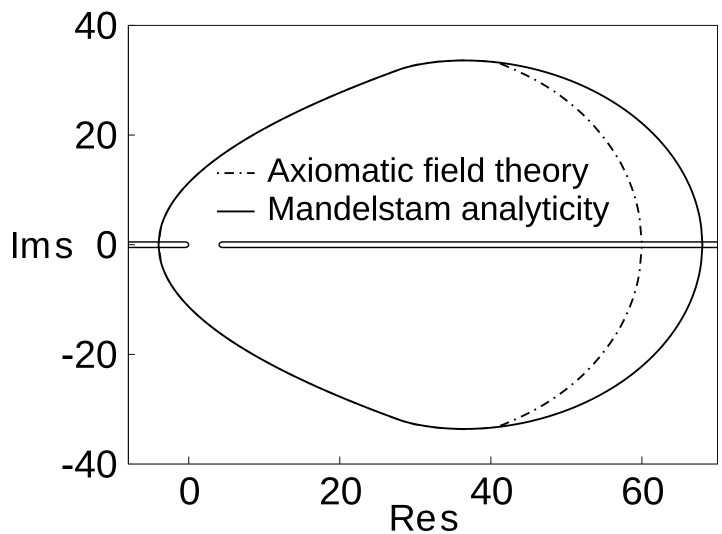

s in units of M_π^2

- Proof is based on first principles, general quantum field theory

A. Martin, *Scattering Theory: Unitarity, Analyticity and Crossing*, Lecture Notes in Physics, vol. 3, 1969.

G. Mahoux, S. M. Roy and G. Wanders, Nucl. Phys. B70 (1974) 297.

$\Rightarrow$ Exact representation for $S_0^0(s)$ in this region
Do not need to parametrize the amplitude

Evaluation of the pole position

- Have an exact formula for the pole position in terms of physical quantities: $S_0^0(s) = 0$
- For the central solution of the Roy equations, $S_0^0(s)$ has two pairs of zeros in the region where the formula holds:

$$s = (6.2 \pm i\,12.3)\,M_\pi^2 \quad \sigma$$
$$s = (51.4 \pm i\,1.4)\,M_\pi^2 \quad f_0(980)$$

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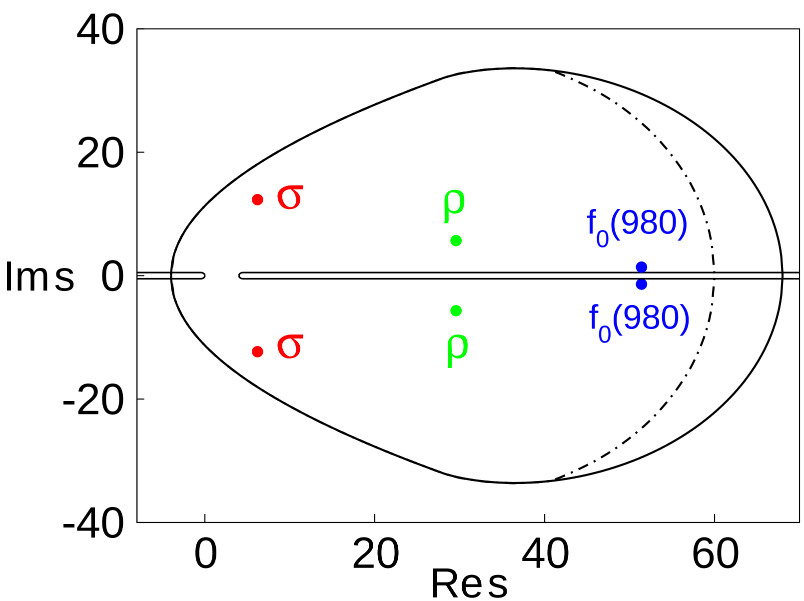


s in units of M_π^2

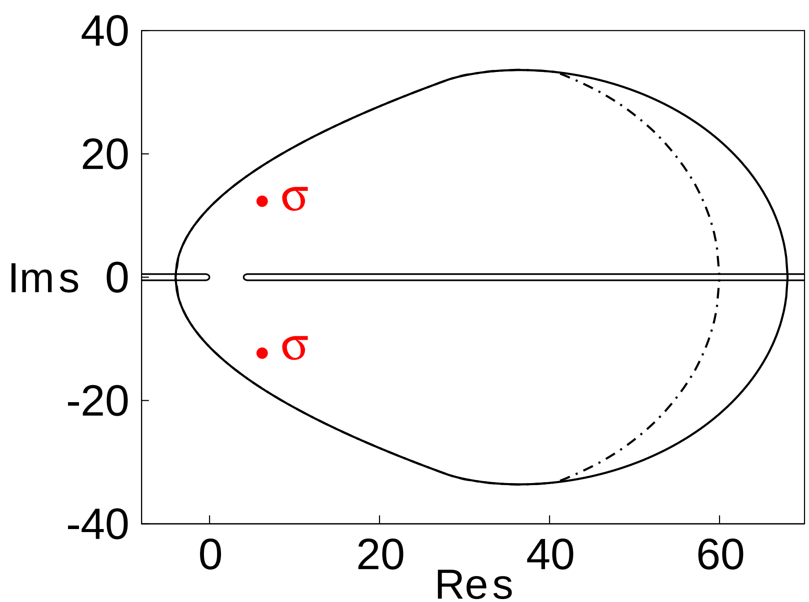
The eyes of the red dragon

Tail at 1.7 GeV: $s \simeq 150\,M_\pi^2$

Result

- Lowest resonance of QCD has vacuum quantum numbers
- Pole on lower half of sheet II occurs in vicinity of

$$m_\sigma = 441 - i\,272\,\text{MeV} = M_\sigma - \tfrac{i}{2}\Gamma_\sigma$$



s in units of M_π^2

Loci Oculorum Draconis Rutili

T. Barnes, Theory summary, MESON 2006

Error analysis

- Result depends on phenomenological input used when solving the Roy equations, subject to uncertainties
  Can follow error propagation explicitly
- Pole position of $f_0(980)$ sensitive to input used for $\eta_0^0(s)$
- Pole position of σ mainly depends on 3 input variables:

  $a_0^0, a_0^2, \delta_A \equiv \delta_0^0$(800 MeV)

  - Information about a_0^0, a_0^2 is in good shape
  - Substantial uncertainties in phenomenology of δ_A
  - Use conservative range: $\delta_A = 82.3^\circ{}^{+10^\circ}_{-4^\circ}$

Error analysis

- Noise from remaining input variables is very small:
  $m_\sigma = (441 \pm 4) - i(272 \pm 6)$ MeV

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  but the values of a_0^0, a_0^2, δ_A are crucial:

$$
\begin{aligned}
m_\sigma = & \,(441 \pm 4) - i(272 \pm 6) \\
& + \;(-2.4 + i\,3.8)\,\tfrac{a_0^0 - 0.22}{0.005} \\
& + \;(0.8 - i\,4.0)\,\tfrac{a_0^2 + 0.0444}{0.001} \\
& + \;(5.3 + i\,3.3)\,\tfrac{\delta_A - 82.3}{3.4}
\end{aligned}
$$

numbers in MeV

Error analysis

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$$m_\sigma = (441 \pm 4) - i(272 \pm 6) \text{ MeV}$$

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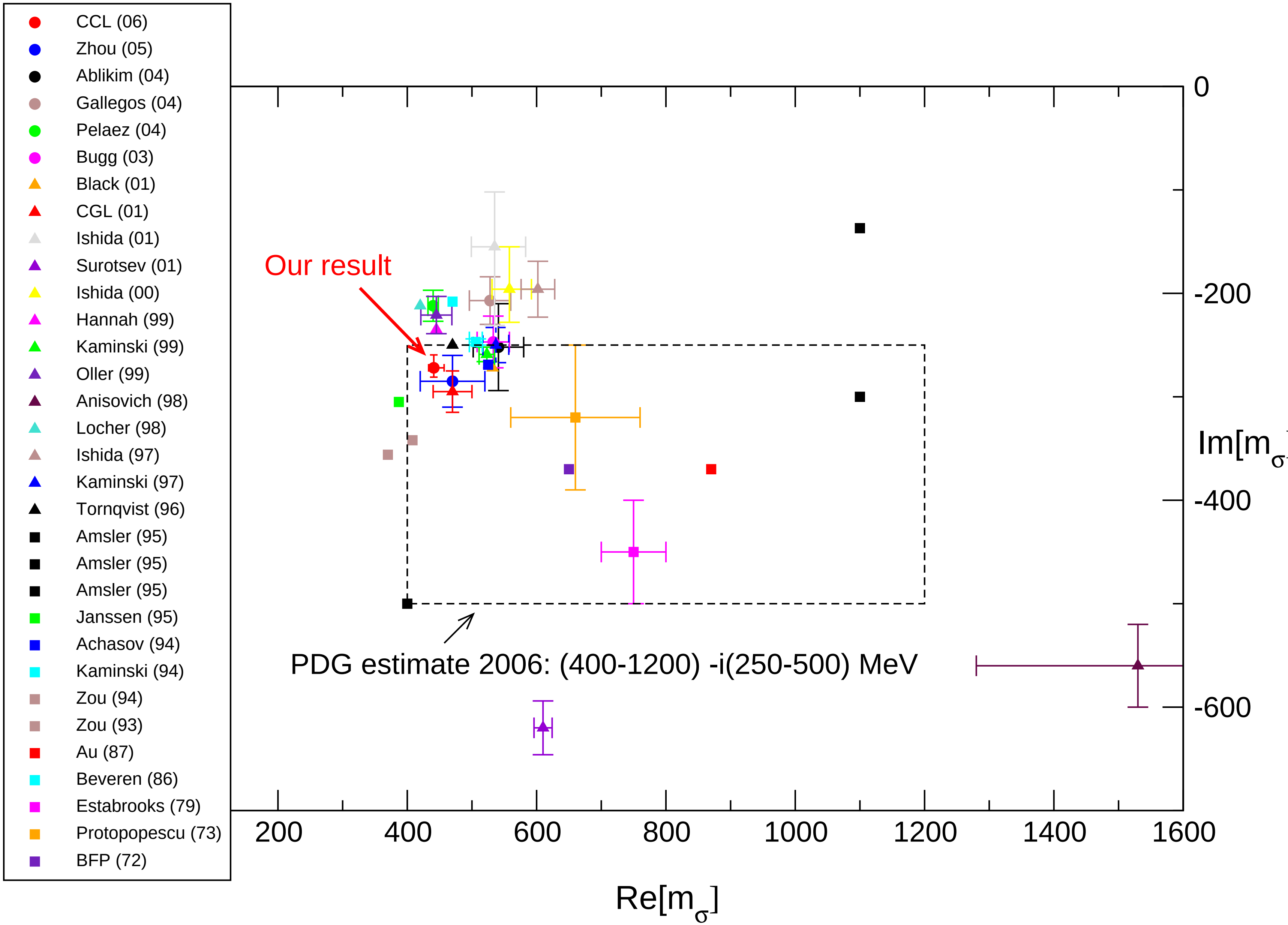
numbers in MeV

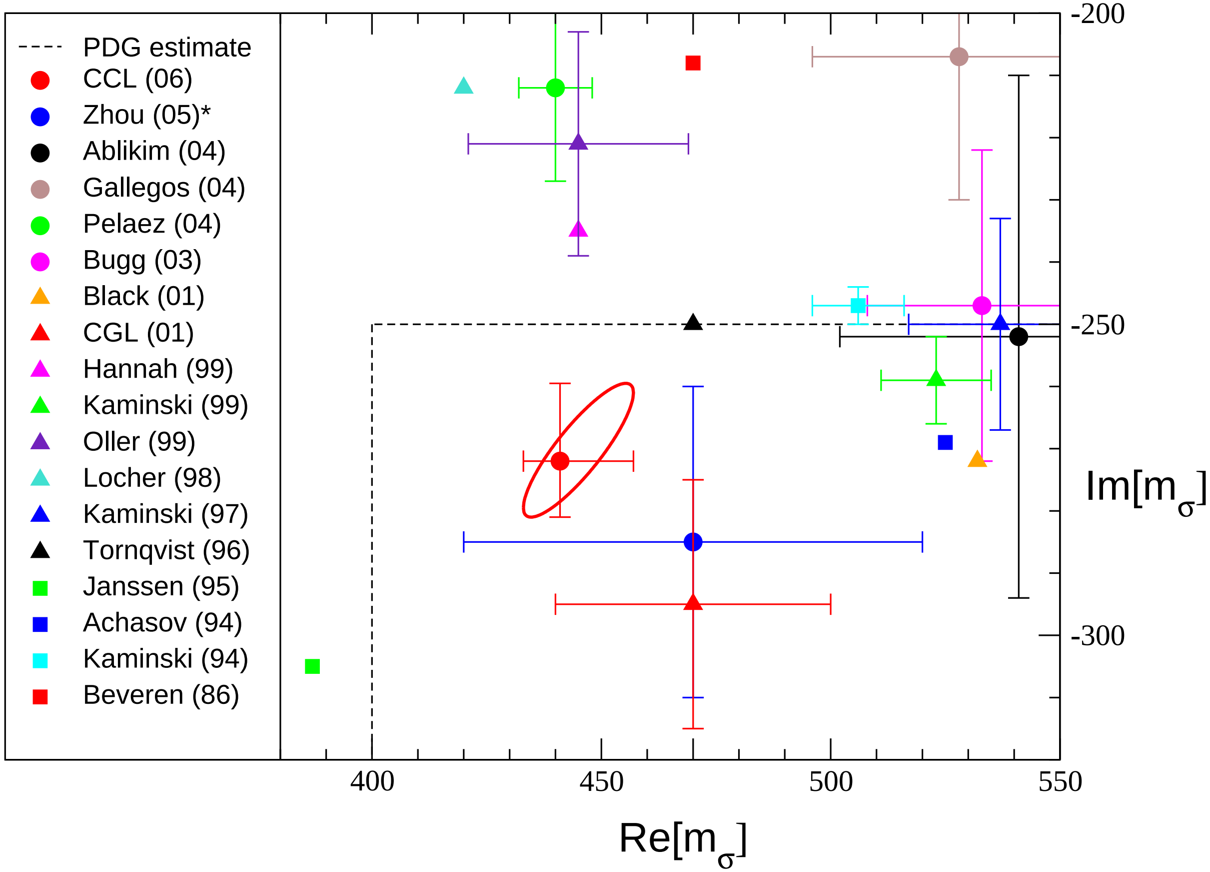
- Final result: insert the predictions for a_0^0, a_0^2, use the phenomenological range for δ_A and add errors up:

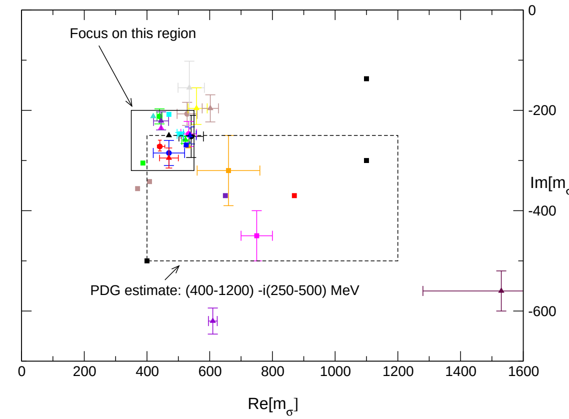
$$m_\sigma = 441\,^{+16}_{-8}\; - i\; 272\,^{+9}_{-13} \text{ MeV}$$

Comparison with compilation of PDG



Vicinity of the pole

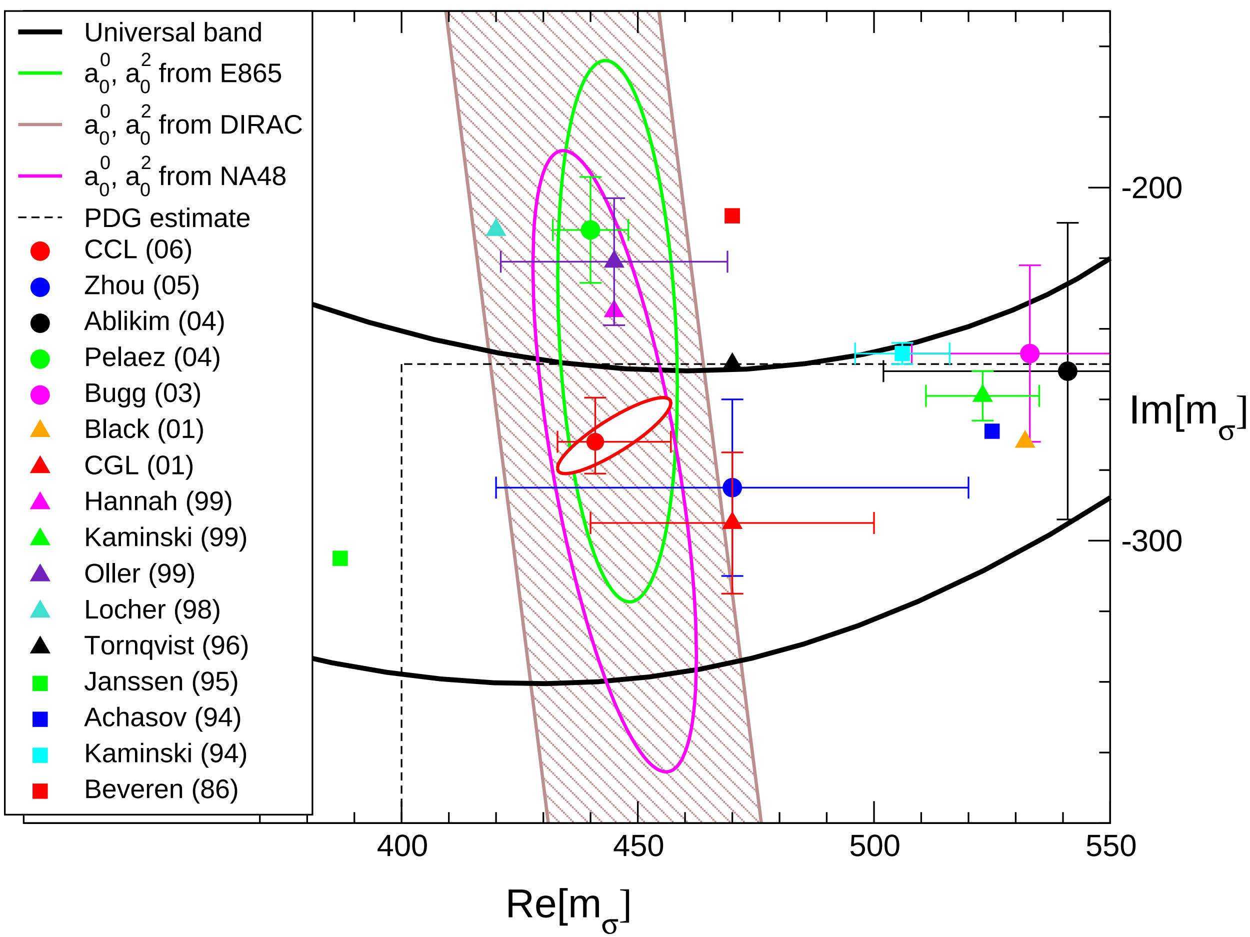




* Zhi-Yong Zhou, Guang-You Qin, Peng Zhang, Zhi-Guang Xiao, Han-Qing Zheng, Department of Physics, Peking University, Beijing &
Ning Wu, Institute for High Energy Physics, Chinese Academy of Science, Beijing

Ignore the theoretical predictions for a_0^0, a_0^2

- Replace the low energy theorems for a_0^0, a_0^2 by the experimental results from E865, DIRAC and NA48
- $a_0^0, a_0^2 \in$ universal band



Why are our errors so incredibly small ?

- The σ occurs at low energies
- At low energies, the subtraction term dominates

$$t_0^0(s) \simeq a_0^0 + (2a_0^0 - 5a_0^2)\frac{(s-4M_\pi^2)}{12M_\pi^2}$$

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$$t_0^0(s) \simeq a_0^0 + (2a_0^0 - 5a_0^2)\frac{(s - 4M_\pi^2)}{12M_\pi^2}$$

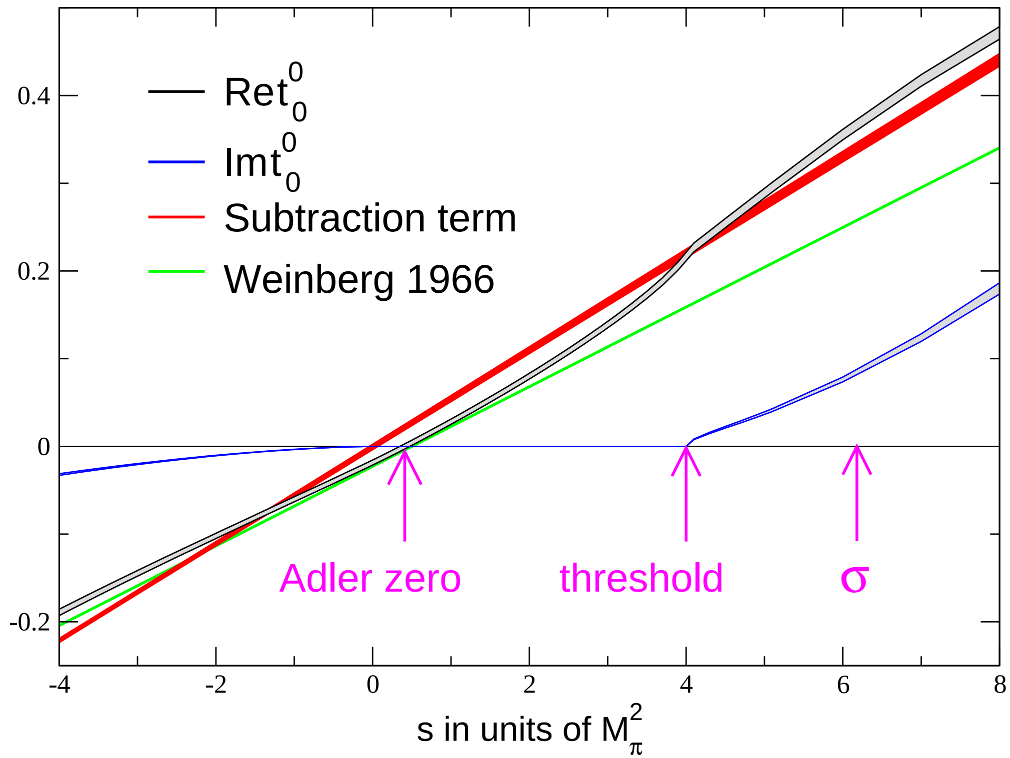
Insert low energy theorem for a_0^0, a_0^2

$\Rightarrow$ Roy equation reduces to Weinberg formula

$$t_0^0(s) \simeq \frac{(2s - M_\pi^2)}{32\pi F_\pi^2}$$

Dispersion integrals only represent a correction

At low energies, the subtraction term dominates



$$s = (0.41 \pm 0.06)\, M_\pi^2$$ Adler zero

$$s = (6.2 - i\, 12.3)\, M_\pi^2$$ pole from σ

Goldstone bosons of low energy interact only weakly

Estimate pole position on back of an envelope

- Approximate $t_0^0(s)$ with the Weinberg formula

$$t_0^0(s) = \frac{(2s - M_\pi^2)}{32\pi F_\pi^2}$$

Where are the zeros of $S_0^0(s)$ in this approximation ?

Estimate pole position on back of an envelope

- Approximate $t_0^0(s)$ with the Weinberg formula

$$t_0^0(s) = \frac{(2s - M_\pi^2)}{32\pi F_\pi^2}$$

Where are the zeros of $S_0^0(s)$ in this approximation ?

$$1 + 2\,i\,\sqrt{1 - 4M_\pi^2/s}\;t_0^0(s) = 0$$

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  For the quantity that counts, the accuracy is modest
- Real zero on sheet II, near $s = 0$ (full amplitude has kinematic singularity: vanishes on sheet II at $s = 0$)

Curvature due to the left hand cut

- Left hand cut generates curvature
  Main contribution on the left stems from the ρ
- Most pole determinations neglect the left hand cut
  Pole from σ is too close for this to be justified
- Can estimate contributions from left hand cut with χPT

  Zhou, Qin, Zhang, Xiao, Zheng, Wu, JHEP 0502 (2005) 043

$\Rightarrow$ Outcome for pole position agrees with our result within the quoted errors

Calculate pole position from phenomenology

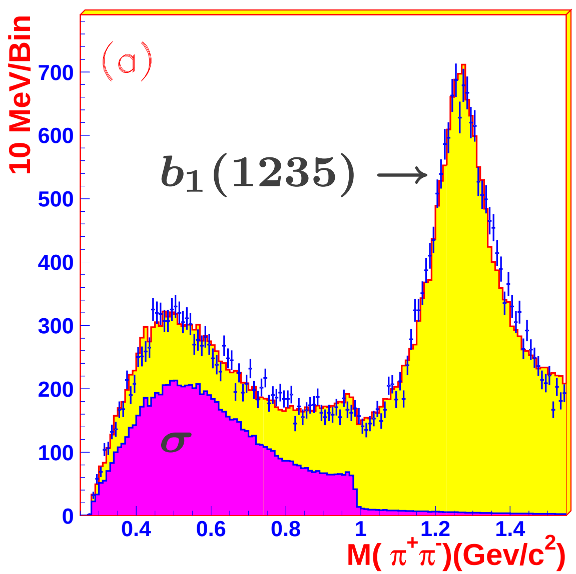
- Ignore the representation of the scattering amplitude obtained from the Roy equations
- Instead use a phenomenological one

J. R. Peláez and F. J. Ynduráin Phys. Rev. D71 (2005) 074016

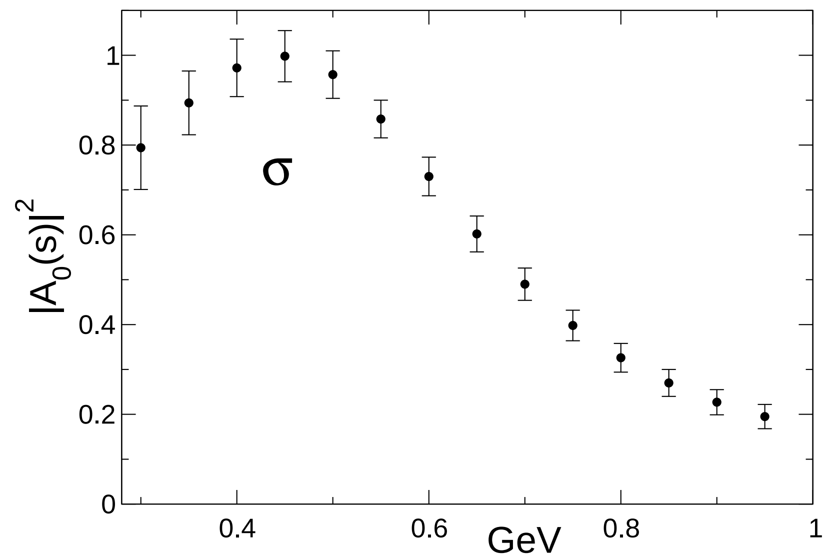
- Insert it in formula for $S_0^0(s)$ and calculate the zeros
  With the central values of PY, this gives
  $m_\sigma = 445 - i\,241$ MeV   to be compared with our central value $m_\sigma = 441 - i\,272$ MeV
- Uncertainties in phenomenology are large
  Those in a_0^0, a_0^2 alone give
  $m_\sigma = (445 \pm 8) - i(241 \pm 22)$ MeV

$\Rightarrow$ Calculation confirms our result within errors

BES data on $J/\psi \to \omega\pi\pi$



BES, Phys. Lett. B598 (2004) 149



S-wave projection (D. Bugg, priv. comm.)

Outcome for pole position:

$m_\sigma = (541 \pm 39) - i\,(252 \pm 42)$ MeV BES 2004

(simple parametrization à la Breit-Wigner, $K\bar{K}$ and $\eta\eta$ final states neglected)

$m_\sigma = (472 \pm 30) - i\,(271 \pm 30)$ MeV Bugg hep-ph/0608081

(reanalysis based on a more complicated model)

Result is model dependent $\Rightarrow$ systematic uncertainty

Analytic continuation by hand

- Most of the results quoted by the PDG are obtained by
(a) parametrizing the data for real values of s
(b) continuing this parametrization analytically in s
- Works well for resonances of modest width: extrapolation then only goes over a short distance and is not very sensitive to the parametrization used

Analytic continuation by hand

- Most of the results quoted by the PDG are obtained by
(a) parametrizing the data for real values of s
(b) continuing this parametrization analytically in s
- Works well for resonances of modest width: extrapolation then only goes over a short distance and is not very sensitive to the parametrization used
- The width of the σ is not small $\Rightarrow$ result for the pole position depends on the parametrization
- Whenever the choice of the parametrization matters, it is better not to use a parametrization

Model independent discussion of $J/\psi \to \omega\pi\pi$

- Neglect rescattering on the ω and 4π final states

$\Rightarrow$ Watson theorem fixes phase of decay amplitude:

$$A_0(s) = |A_0(s)|\, e^{i\delta_0^0(s)} \quad \text{for } 4M_\pi^2 < s < 4M_K^2$$

$\uparrow$ I. Caprini, Phys. Lett. B638 (2006) 468

S-wave projection of decay amplitude

- Situation is the same as for the scalar form factor

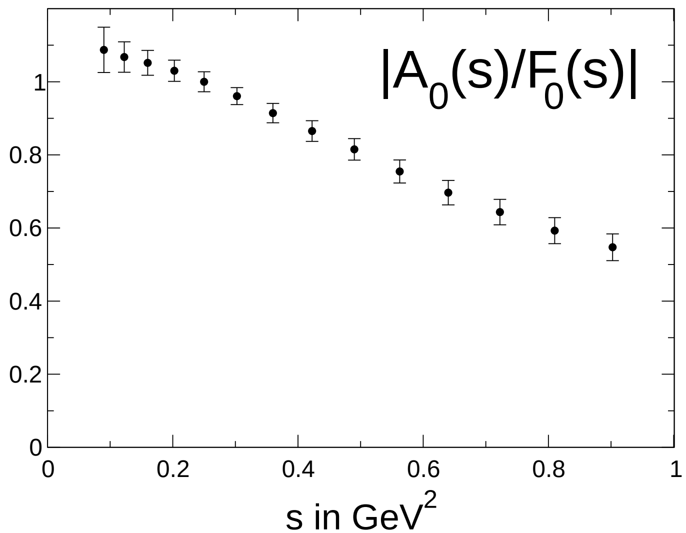
$$F_0(s) = \langle \pi\pi \,\text{out}\,|\,\bar{u}u\,|0\rangle = |F_0(s)|\, e^{i\delta_0^0(s)}$$

$\Rightarrow$ $A_0(s)/F_0(s)$ is real for $0 < s < 4M_K^2$

- Both $A_0(s)$ and $F_0(s)$ have a pole from the σ on the second sheet, drops out in $A_0(s)/F_0(s)$
- r.h. cut in $A_0(s)/F_0(s)$ only starts at $4M_K^2$

$\Rightarrow$ $A_0(s)/F_0(s)$ can vary only slowly with s

Comparison with scalar form factor



- $F_0(s)$ taken from Ananthanarayan et al. (2004), based on central solution of the Roy equations
- Model of Lähde and Meißner, hep-ph/0606133 describes J/ψ decays into $\omega\pi\pi, \omega K\bar{K}, \phi\pi\pi$, $\phi K\bar{K}$ in terms of scalar form factors, uses crude approximation: $A_0(s)/F_0(s) \simeq$ constant

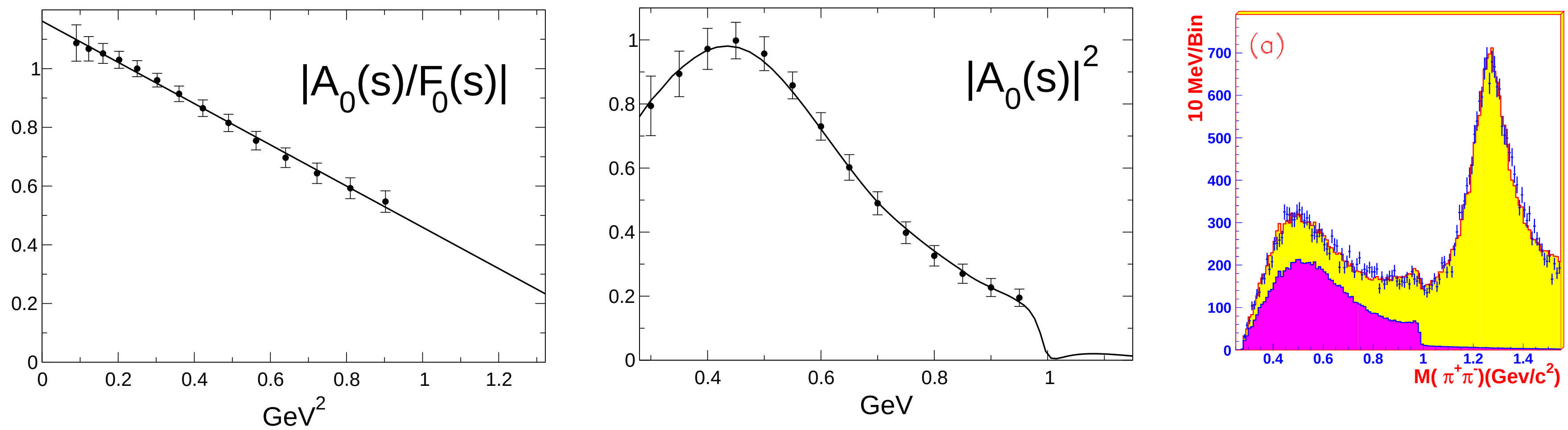
- Dispersion relation for $R(s) \equiv A_0(s)/F_0(s)$:

$$R(s) = R_0 + R_1\, s + \frac{(s-2M_K^2)^2}{\pi} \int \frac{dx\, \mathrm{Im} R(x)}{(x-2M_K^2)^2\,(x-s)}$$

- Plot does not show any curvature $\Rightarrow$ integral is small

$$R(s) \simeq R_0 + R_1\, s$$

Comparison of $J/\psi \to \omega\pi\pi$ and scalar form factor



- Full line corresponds to the approximation

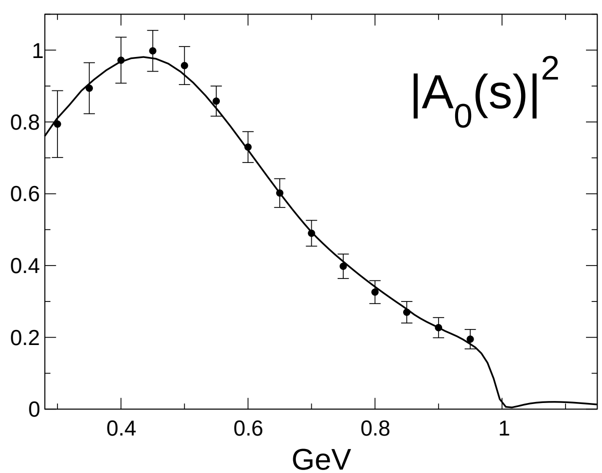
  $A_0(s) \simeq R_0\,(1 - s/s_0)\,F_0(s)$, with $s_0 = 1.65$ GeV^2

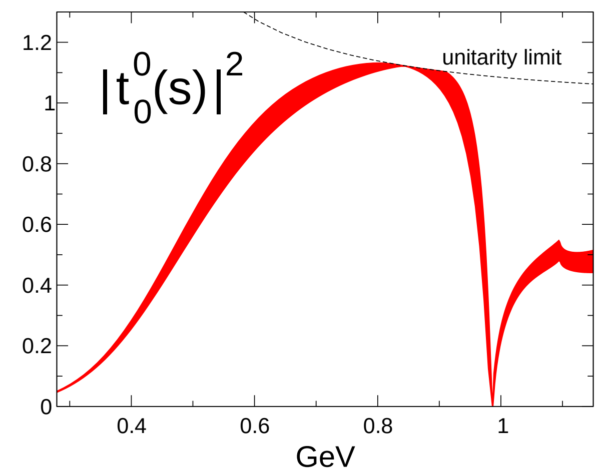
- Observed energy dependence is consistent with the assumption that rescattering on the ω can be neglected

- In contrast to the case of $\pi\pi$ scattering, chiral symmetry does not determine the two subtraction constants occurring here

Comparison of $J/\psi \to \omega\pi\pi$ and $\pi\pi$ scattering

- $A_0(s)$ and $t_0^0(s)$ have approximately the same phase but profile is not the same: Adler zero in $t_0^0(s)$





- Need two subtractions – these make the difference
- Data on $J/\psi \to \omega\pi\pi$ are better
  Theory is weaker (unitarity, subtractions, rescattering)

$\Rightarrow$ Uncertainty in pole position from $J/\psi \to \omega\pi\pi$ larger

L., in Proc. MESON 2006, hep-ph/0608218

Further comments on models: unitarization of χPT

- $\pi\pi$ scattering amplitude known to two loops of χPT

  Bijnens, Colangelo, Ecker, Gasser & Sainio 1996

- χPT works very well below threshold, but goes out of control long before the energy reaches M_ρ
- Range of validity of χPT can be extended by hand: "Unitarized χPT", "Inverse Amplitude Method"
- Padé: unitarity $\surd$ poles from ρ, σ $\surd$

  Truong, Dobado, Herrero, Peláez, Hannah, Oller, Guerrero, Ramos, Oset, Ang, Xiao, Zheng, Song, He, Qin, Deng, Nieves, Pavón Valderrama, Ruiz-Arriola, Gómez-Nicola, Llanes-Estrada, Lähde, Meissner, . . .

- Simple, useful approximation, also for form factors
  Improves chiral representation in physical region

Caveats with unitarization models

- Cannot solve unitarity & crossing symmetry by hand
  IAM enforces unitarity at the expense of crossing
  - IAM with one loop χPT fails near Adler zero
  - Left hand cut is deformed near $s = 0$
  - Fake poles ($I = 2$!)
- Thorough discussion of these problems
  - Dobado & Peláez 1993
  - Ang, Xiao, Zheng & Song 2001
  - Qin, Deng, Xiao & Zheng 2002
- IAM is a model $\Rightarrow$ not clear how to estimate the uncertainties to be attached to the results

Physical interpretation of the σ

- Glueball ? $\bar{q}q$? $\bar{q}q\bar{q}q$?
- Hadrons in terms of quarks and gluons ?
- Fock space is basis of perturbation theory – not clear how to use it quantitatively in nonperturbative domain

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- My qualitative picture of the σ:
  - Can understand position of σ pole in terms of F_π

$\Rightarrow$ Low energy properties of the pions are relevant

$\Rightarrow$ Physics of the $\sigma \in$ Goldstone boson dynamics

$\Rightarrow$ Head of the dragon contains only little glue

$\Rightarrow$ Wave function has large tetra-quark component

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  $\Rightarrow$ Head of the dragon contains only little glue

  $\Rightarrow$ Wave function has large tetra-quark component
- This picture is by no means commonly accepted

  Törnqvist, Ishida, Jaffe, Minkowski, Ochs, Bugg, Pennington, Peláez, Oller, Hannah, Guo, Su, Xiao, Zheng, Zhou, Chen, Hosaka, Zhu, Liu, Maiani, Polosa, Piccinini, Riquer, Isidori, Nicolaci, Pacetti, Menessier, Narison, Fariborz, Jora, Schechter, van Beveren, Kleefeld, Rupp, Scadron, Ynduráin, García-Martín, . . .

$\Rightarrow$ Comprehensive review : Klempt & Zaitsev, arXiv:0708.4016

$\Rightarrow$ Talks by N. Wu, H. X. Chen, Z. H. Guo

The κ

- $K\pi$ scattering amplitude obeys an analog of the Roy equations. Pole from κ can be calculated on this basis

$$m_\kappa = (658 \pm 13) - i\,(278.5 \pm 12)\,\text{MeV}$$

Descotes-Genon and Moussallam 2006

- Confirms an earlier calculation, where the l.h. cut was estimated with χPT Zhou and Zheng 2006

- Back-of-the-envelope calculation for $K\pi$ gives

$$m_\kappa = 671 - i\,262\,\text{MeV}$$

⇒ Physics of σ and κ is very similar

Qualitative picture for κ, $f_0(980)$, ...

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- $f_0(980)$ and $a_0(980)$
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  - Interaction among two kaons plays important role

Hanhart 2007

$\Rightarrow$ Talk by B. S. Zou

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Hanhart 2007

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- Multiplet pattern ?
  - The $\pi\pi$, πK, $K\bar{K}$ thresholds strongly break SU(3)

  $\Rightarrow$ Expect strong symmetry breaking in the masses and widths of the lowest 0^+ resonances
  - Do these form complete SU(3) multiplets at all ?

Remark on $K\pi$ scattering

- 2 subtraction constants, dominate at low energies: $a_0^{\frac{1}{2}}$ (positive), $a_0^{\frac{3}{2}}$ (negative, small) $\leftrightarrow$ a_0^0, a_0^2 predictions less accurate: rely on expansion in m_s
- SU(2)×SU(2) theorem for $a_0^- = \frac{1}{3}(a_0^{\frac{1}{2}} - a_0^{\frac{3}{2}})$:

$$a_0^- = \frac{M_\pi^2}{8\pi F_\pi^2(1 + M_\pi/M_K)}\{1 + O(M_\pi^2)\}$$

$$\text{compare } \pi\pi : \quad a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2}\{1 + O(M_\pi^2)\}$$

- Final state interaction in $K\pi$ weaker than in $\pi\pi$

$\Rightarrow$ Corrections for a_0^- should be even smaller than for a_0^0

- Indeed, one loop correction in a_0^- is 12% [a_0^0: 25%]

Roessl 1999, Kubis & Meissner 2002

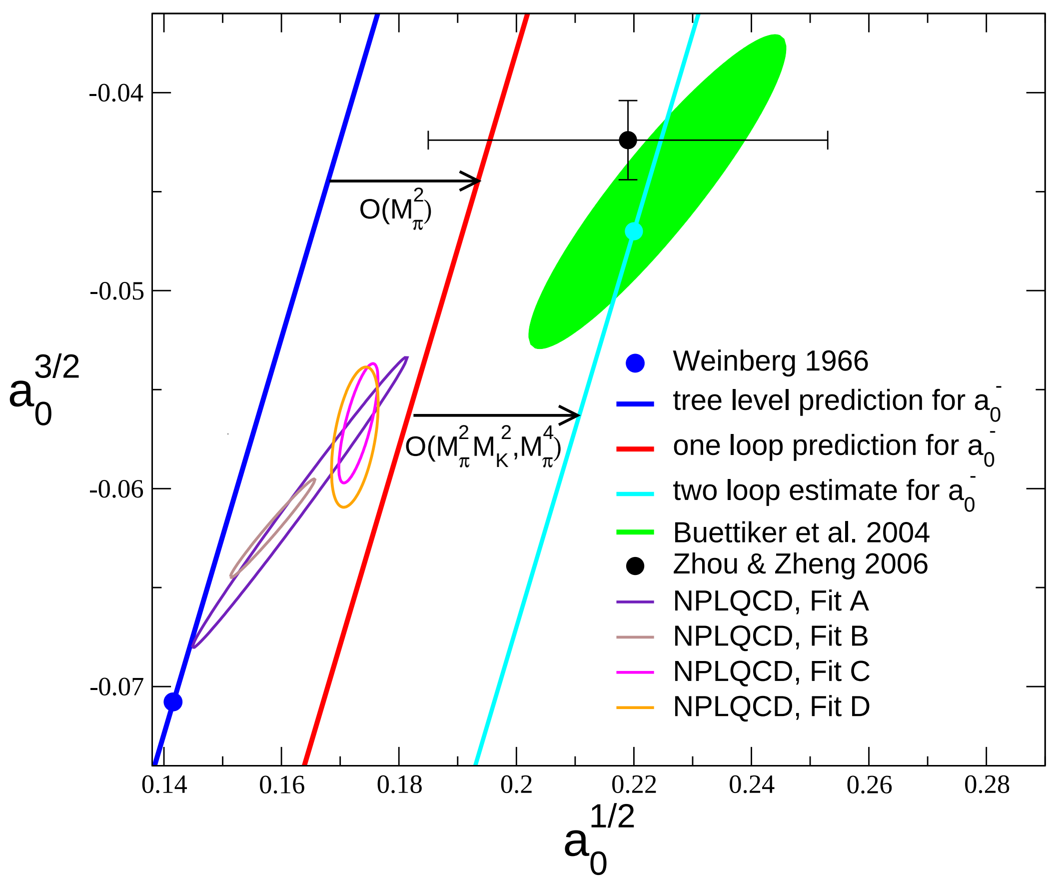
Puzzle

- Phenomenological analysis based on Roy-Steiner does not agree well with the one loop prediction for a_0^-

  Büttiker, Descotes-Genon & Moussallam 2004

- Estimate for the $O(p^6)$ couplings gives large correction

  Bijnens, Dhonte & Talavera 2004, Schweizer 2005, Kaiser & Schweizer 2006



?

Need to solve the puzzle

- Does the expansion in powers of momenta fail already at threshold, because $M_K + M_\pi > 2M_\pi$?

$\Rightarrow$ If so, fix the subtractions at $s = u$, $t = 2M_\pi^2$

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- First lattice result for a_0^- is between tree and one loop results of χPT, but needs confirmation

NPLQCD, hep-lat/0607036

a_0^- can be measured by means of $K\pi$ atoms
Is there a reliable prediction and if so, what is it ?

Conclusion

- Low energy pion physics: theory ahead of experiment
  - Precision experiments carried out and under way
  - Lattice makes slow, but steady progress
  - So far, all tests confirm the theory

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- Low energy pion physics: theory ahead of experiment
  - Precision experiments carried out and under way
  - Lattice makes slow, but steady progress
  - So far, all tests confirm the theory
- Limitations of our approach:
  - Calculations cannot be done on back of an envelope
  - Analysis only covers low energies
    Extension to higher energies is under way
  - Only a few applications have been worked out:
    $\pi\pi$ scattering, pion form factors, hadronic vacuum polarization in muon $g-2$
  - Much is yet to be done: $J/\psi \to \omega\pi\pi$, $D \to 3\pi$,
    $\gamma\gamma \to \pi\pi$, πK, πN, $\ldots$

Conclusion

- Model independent method for analytic continuation
  - The lowest resonance of QCD occurs at

    $M_\sigma = 441\,^{+16}_{-8}$ MeV $\qquad \Gamma_\sigma = 544\,^{+18}_{-25}$ MeV

    and carries vacuum quantum numbers
  - Crossing symmetry plays an essential role:
    Fixes contributions from left hand cut
    Ensures fast convergence, low energy dominance
  - Pole occurs at low value of s, closer to left hand cut than to singularities from $K\bar{K}$, $f_0(980)$
  - Result for Γ_σ relies on theory for a_0^2
    Experiments concerning a_0^2 would be most welcome

VISIT THE RED DRAGON

GENTLE ANIMAL

LOOK IN HIS EYES FROM CLOSE

SMELL HIS GOOD BREATH

BRING YOUR PIONS ALONG AND

FEED HIM YOURSELF

The management denies responsibility for incidents involving the dragon's tail

SPARES

Causality

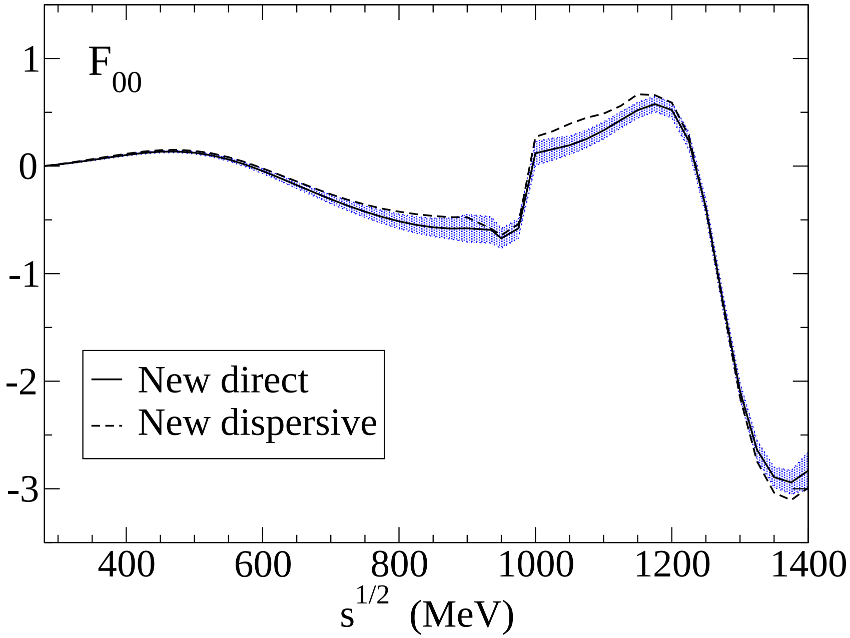
- Causality strongly constrains the scattering amplitude
- Prototype: forward dispersion relation
  Can be used to improve parametrizations of the data
- Example: $\pi^0\pi^0 \to \pi^0\pi^0$

  $$F_{00}(s,t) = N\{T^0(s,t) + 2\,T^2(s,t)\}$$
- Forward amplitude: $\mathrm{Im}F_{00}(s,0) \sim \sigma_{\text{tot}}^{\pi^0\pi^0}$
- Forward dispersion relation:

  $$F_{00}(s,0) = 32\pi N(a_0^0 + 2a_0^2) + \frac{2s(s-4M_\pi^2)}{\pi}\int_{4M_\pi^2}^{\infty}\frac{(x-2M_\pi^2)\,\mathrm{Im}F_{00}(x,0)\,dx}{x(x-4M_\pi^2)(x-s)(x+s-4M_\pi^2)}$$

$\Rightarrow$ Causality intertwines low and high energies

Forward dispersion relation for $\pi^0\pi^0 \longrightarrow \pi^0\pi^0$



Kamiński, Peláez & Yndúráin 2006

Use dispersion relation to improve PWA

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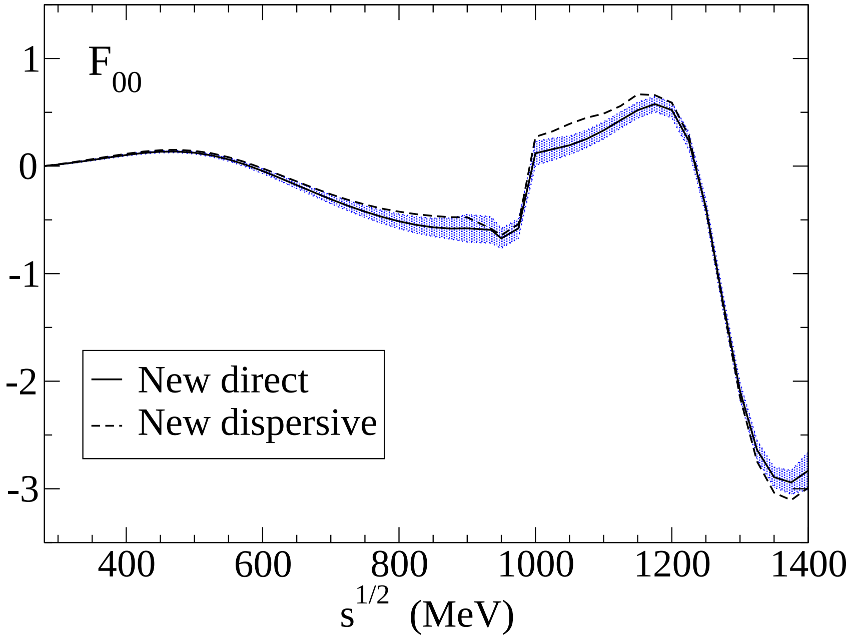

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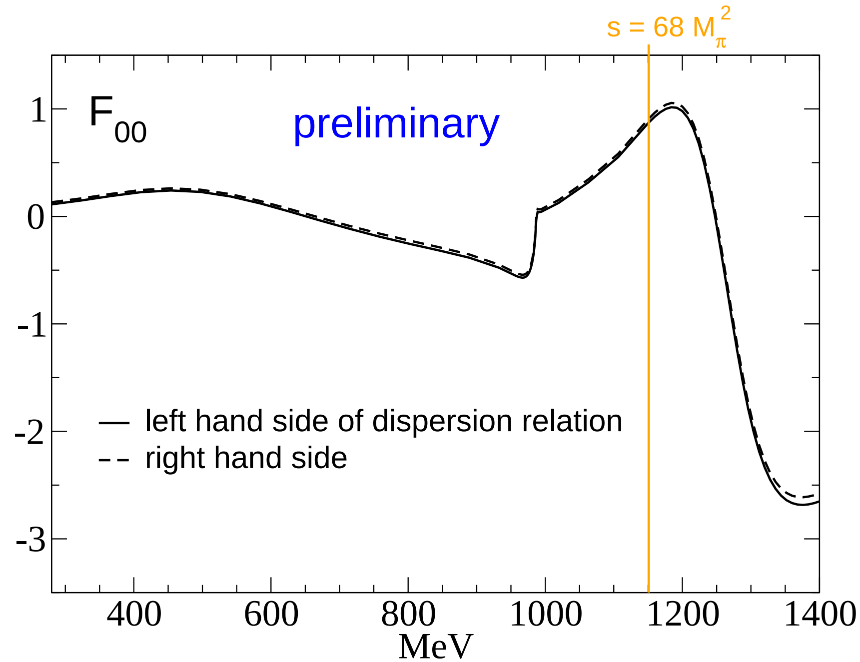


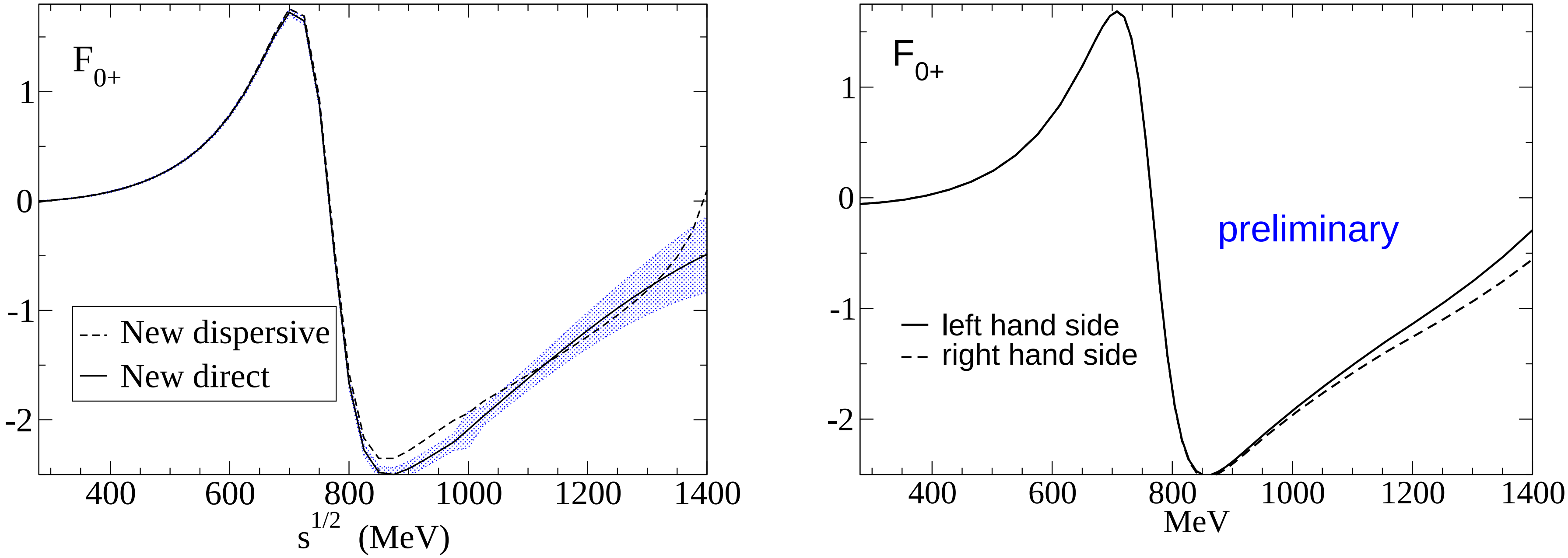
Caprini, Colangelo & L., central solution
of Roy equations for S- & D-waves

$$F_{00}(s,0)=32\pi N\{t_0^0(s)+2\,t_0^2(s)+5\,t_2^0(s)+10\,t_2^2(s)+\ldots\}$$

If the partial waves obey the Roy equations, then the sum over <u>all</u> of these automatically satisfi es forward dispersion relations

Roy equations only hold for $s < 68M_\pi^2$, but this limitation does not appear to be essential

Forward dispersion relation for $\pi^0\pi^+ \longrightarrow \pi^0\pi^+$



Kamiński, Peláez & Ynduráin 2006
Use dispersion relation to improve PWA

Caprini, Colangelo & L.
Solve Roy equations for S-, P- & D-waves

$$F_{0+}(s,0) = 32\pi N \{ t_0^2(s) + 3\, t_1^1(s) + 5\, t_2^2(s) + 7\, t_3^1(s) + \ldots \}$$