Orsay, November 2005

How well do we understand the interaction among the pions at low energies ?

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$\pi\pi$ interaction at low energies

- Plays a crucial role whenever the strong interaction is involved at low energies
- Main experiments on $\pi\pi$ scattering were done in the seventies. What's new ?
- Significant theoretical progress, based on ChPT + dispersion theory

  G. Colangelo, *Introduction to ChPT*, Schleching 2003
- In the isospin limit, the scattering amplitude is characterized by a single function $A(s,t)$
  At low energies, this function can now be predicted to a remarkable degree of precision
- Theory passed one successful test:
  precision data of E865 on $K \to \pi\,\pi\,\ell\,\nu$ are in excellent agreement with the predictions

Comment on isospin breaking

- $A(s,t)$ only exists in a theoretical world where isospin is conserved. In reality, isospin is broken: $M_{\pi^0} \neq M_{\pi^+}$ etc.
- Our analysis is done at $m_u = m_d$, $e = 0$

  F_π = physical value of $F_{\pi^+} \Rightarrow \Lambda_{\rm QCD}$

  M_π = physical value of $M_{\pi^+} \Rightarrow m_u$

  M_K = physical value of $M_{K^+} \Rightarrow m_s$

  m_c, m_b, m_t = physical values
- Can establish contact with experiment only to the extent that isospin breaking is understood

  Cirigliano, Ecker, Neufeld, Pich

  $M_{\pi^+} - M_{\pi^0}$, $M_{K^+} - M_{K^0}$ $\surd$

  π^0-η-mixing, ρ-ω-mixing $\surd$

  $M_{\rho^+} - M_{\rho^0}$, $\Gamma_{\rho^+} - \Gamma_{\rho^0}$?

  Ghozzi & Jegerlehner, Davier

Chiral symmetry

- Goldstone bosons of zero momentum
  do not interact $\rightarrow$ $A(s,t)$ has an Adler zero
- Chiral expansion starts at $O(p^2)$

$$A(s,t) = \frac{s - M_\pi^2}{F_\pi^2} + O(p^4)$$

Weinberg 1966

- Expression is linear in s, t

$\Rightarrow$ only S- and P-waves present at $O(p^2)$

- Representation for $A(s,t)$ known to two loops,
  i.e. up to and including $O(p^6)$

  Bijnens, Colangelo, Ecker, Gasser, Sainio 1996

- Representation is very accurate near the center of the Mandelstam triangle
- The singularities required by unitarity generate curvature, uncertainties grow with the distance from the center of the triangle

  Already at threshold (scattering lengths), the chiral representation leaves to be desired

Roy equations

- ChPT is not needed for dependence on s, t
  Analyticity, unitarity and crossing determine the amplitude in terms of its imaginary part, except for the subtraction constants

- $\pi\pi$ scattering is special: crossed channels are identical

$\Rightarrow$ Re$A(s,t)$ can be represented as an integral over physical region imaginary part

S.M. Roy 1971

- Representation involves 2 subtraction constants, can identify these with 2 scattering lengths:

  $a_0^0 \,,\; a_0^2$ ← isospin
  ← angular momentum

- Representation leads to dispersion relations for the individual partial waves: *Roy equations*

- Roy equations were studied long ago

  early work reviewed by Pennington, Ann.Phys. 1975

  Main problem at that time: experimental information near threshold is meagre

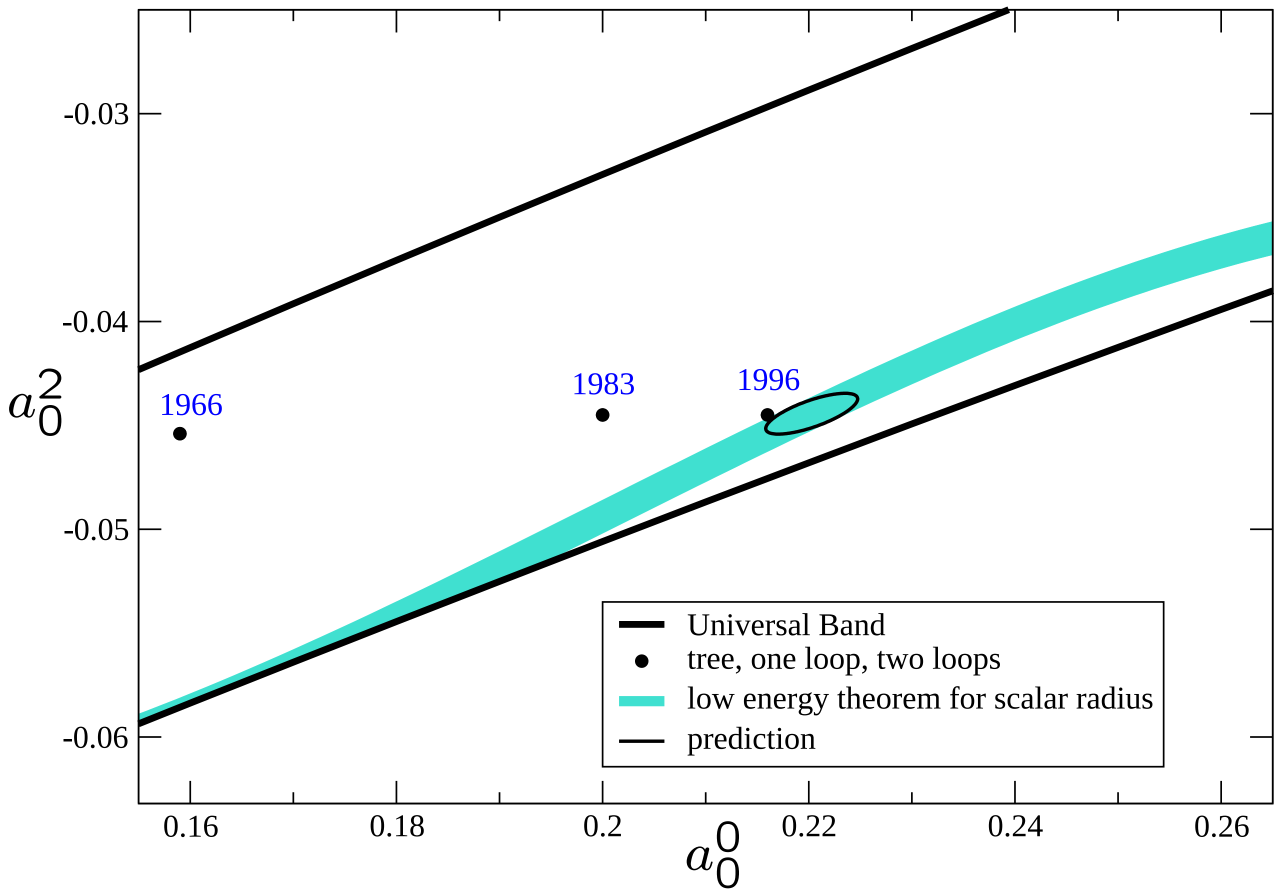
⇒ Large uncertainties in a_0^0, a_0^2

- The two subtraction constants are the essential parameters in the low energy region: given a_0^0, a_0^2, the scattering amplitude can be calculated to within very small uncertainties

  Ananthanarayan, Colangelo, Gasser & L. 2001
  Descotes, Fuchs, Girlanda & Stern 2002

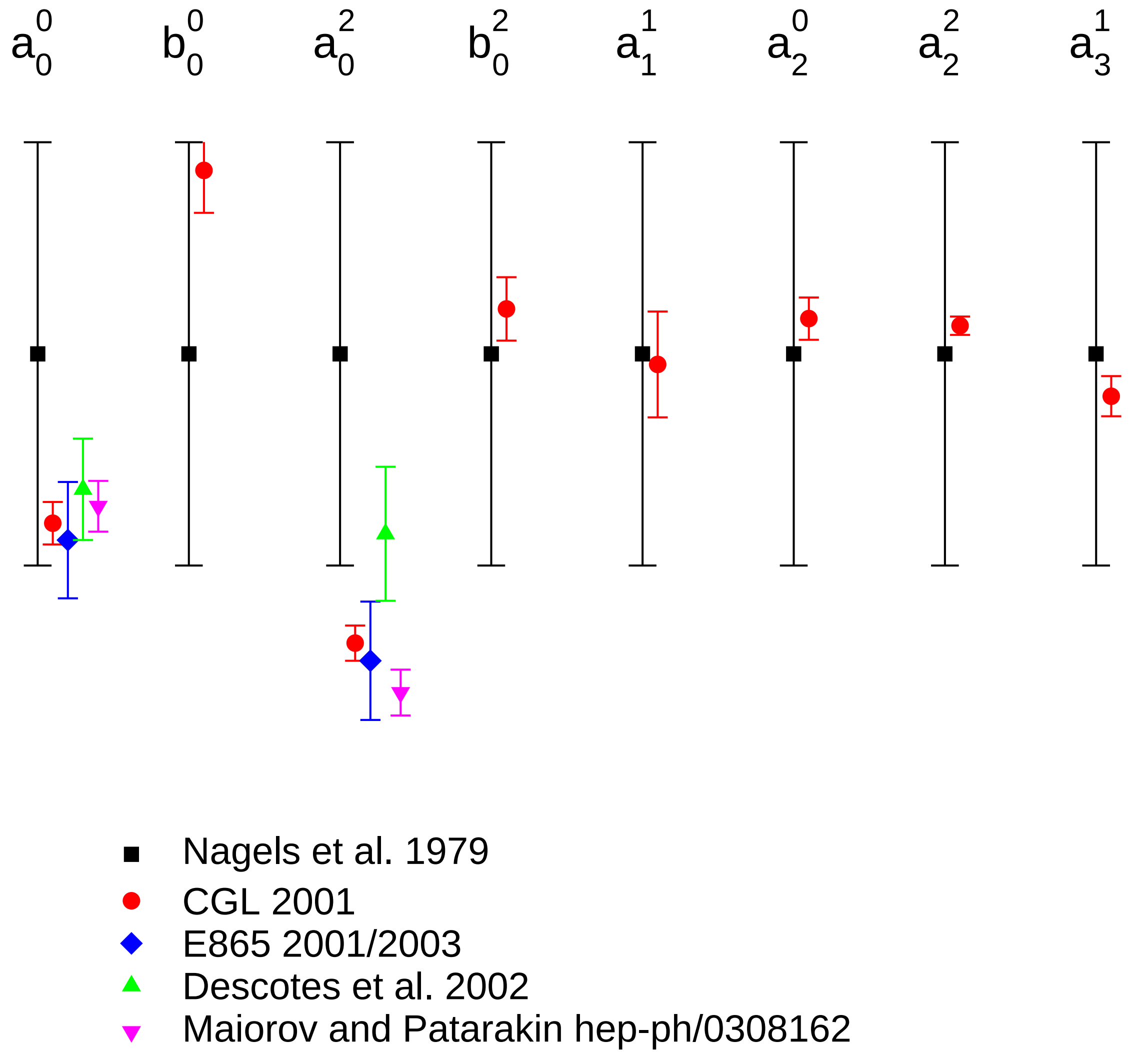
- ChPT provides the missing piece: low energy theorems for a_0^0, a_0^2

- More accurate method: match the two loop and dispersive representations below threshold



Predictions for the S-wave $\pi\pi$ scattering lengths

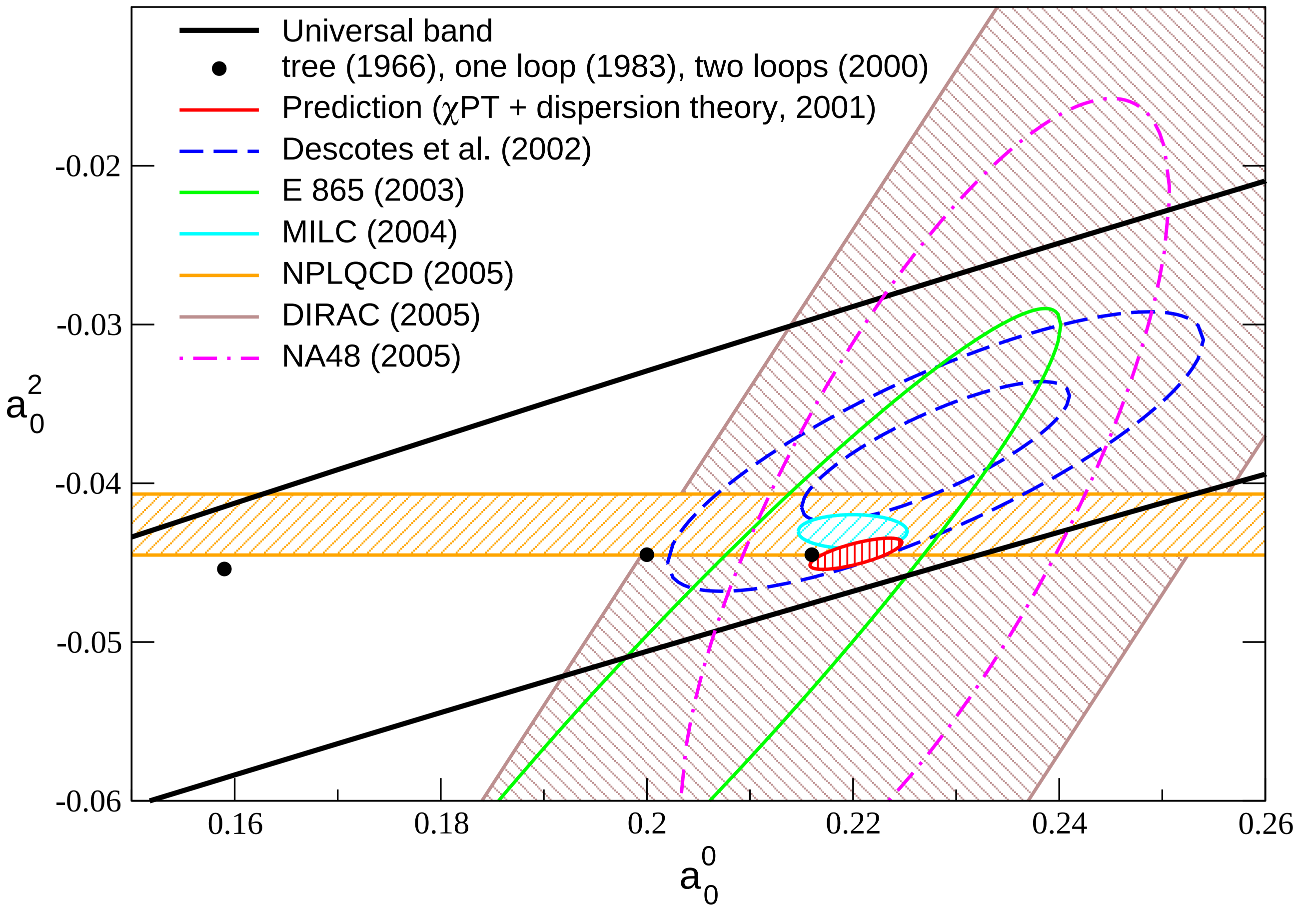
- What difference does it make whether or not the subtraction constants are known ?



⇒ Quantum jump in low energy pion physics

In combination with the low energy theorems of ChPT , the dispersion relations for the partial waves fix the $\pi\pi$ scattering amplitude to a very high degree of accuracy

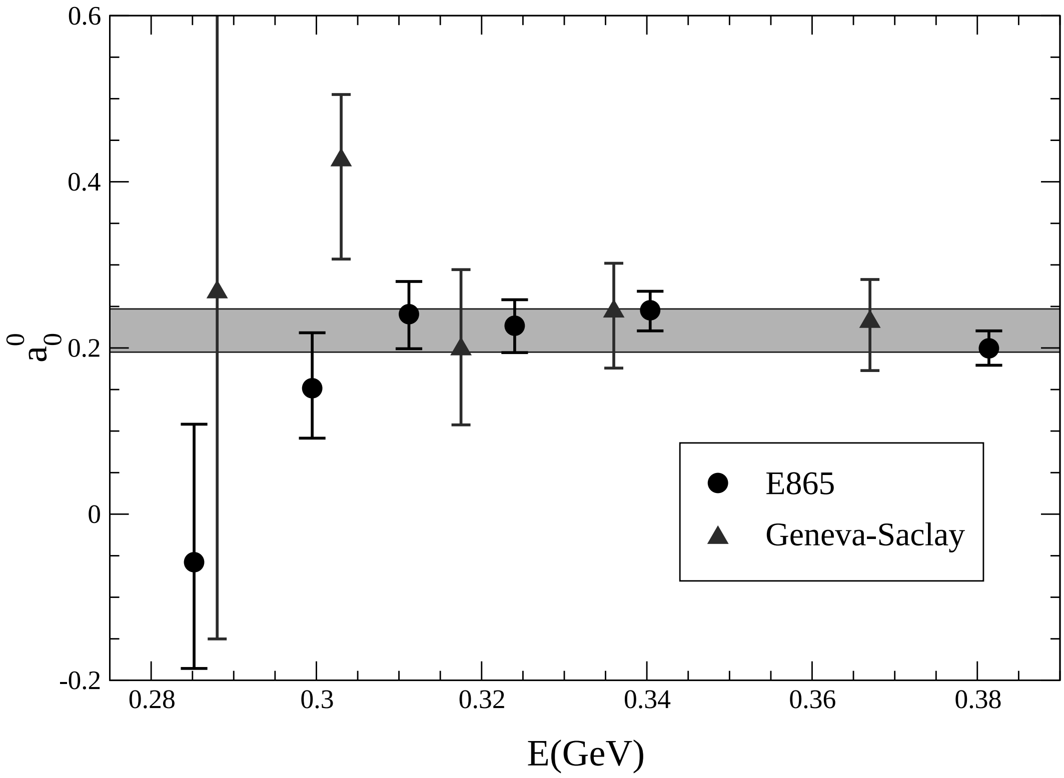
- For once in strong interaction physics, theory is ahead of experiment . . .



State of knowledge of the S-wave $\pi\pi$ scattering lengths

Experimental test via $K \to \pi\,\pi\,e\,\nu$

- New data from E865-collaboration at Brookhaven allow a significant test of the GMOR relation
- Final state interaction theorem: phase of the $K \to \pi\pi e\nu$ transition form factors is determined by elastic $\pi\pi$ scattering amplitude
- Conversely, can measure the phase difference $\delta_0^0 - \delta_1^1$ by means of this decay



- Fit to the data yields

$$\boxed{a_0^0 = 0.216 \pm 0.013\,(\mathrm{stat}) \pm 0.002\,(\mathrm{syst}) \pm 0.002\,(\mathrm{th})}$$

S. Pislak et al., Phys. Rev. D67 (2003) 072004

- To be compared with the prediction of ChPT

$$\boxed{a_0^0 = 0.220 \pm 0.005}$$

Amoros, Bijnens & Talavera, Nucl.Phys. 2000

Colangelo, Gasser & L., Phys.Lett. 2000

- The prediction only holds if the quark condensate is the leading order parameter

⇒ More than 94 % of the pion mass originates in the quark condensate term

$$M_\pi^2 \simeq (m_u + m_d) \times |\langle 0|\,\bar{q}\,q\,|0\rangle| \times \frac{1}{F_\pi^2} \;\surd$$

- Data analysis relies on LET for $\langle r^2 \rangle_s$
  Very important to test that prediction as well
  Descotes, Fuchs, Girlanda & Stern

- Dependence on m_s, Zweig-rule violations ?
  Ananthanarayan, Büttiker, Descotes-Genon, Fuchs, Girlanda, Jamin, Knecht, Moussallam, Oller, Pich, Stern, . . .

- Forthcoming: more precise data on $K \to \pi\,\pi\,e\,\nu_e$ from NA48/2 (CERN) and KLOE (Frascati)

- $\pi^+\pi^-$ atoms provide an ideal laboratory
- Atoms decay through the strong interaction

$$\pi^+\pi^- \longrightarrow \pi^0\pi^0$$

Decay rate $\propto (a_0^0 - a_0^2)^2$

- Interference of e.m. and strong interactions in bound state and decay is now well understood

$\Rightarrow$ Can reliably measure low energy properties of the $\pi\pi$ scattering amplitude in this way

- Prediction for the lifetime: $\tau = 2.9 \pm 0.1$ fs

Gasser, Lyubovitskij, Rusetsky & Gall 2001

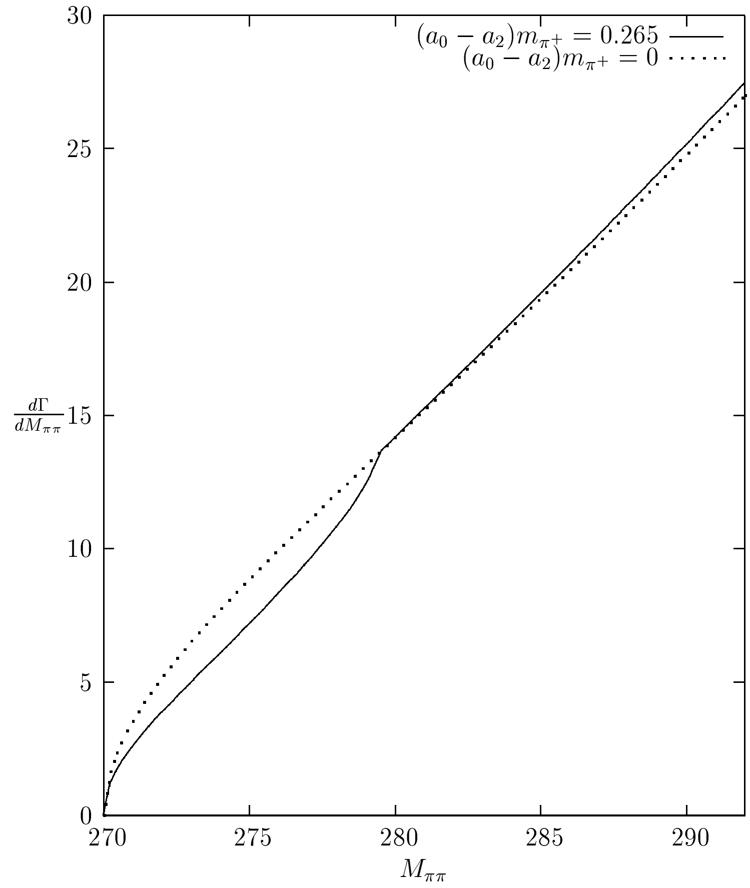
- DIRAC: beautiful experiment done at CERN Aims at a measurement of the lifetime to 10%

$\Rightarrow$ clean test of symmetry breaking due to m_u, m_d

- Using a subset of the collected data, DIRAC has achieved an accuracy of 16%:

$$\tau = 2.85 \,^{+0.48}_{-0.41} \text{ fs}$$

L.Tauscher at www.lnf.infn.it/conference/dafne04/



Taken from N. Cabibbo, hep-ph/0405001

- New idea: accurate data in the threshold region of the decay $K^+ \to \pi^+\pi^0\pi^0$ allow a determination of $a_0^0 - a_0^2$

- Preliminary result of NA48:

  $a_0^0 - a_0^2 =$

  $0.264 \pm 0.006(\text{stat}) \pm 0.004(\text{syst}) \pm 0.013(\text{ext})$

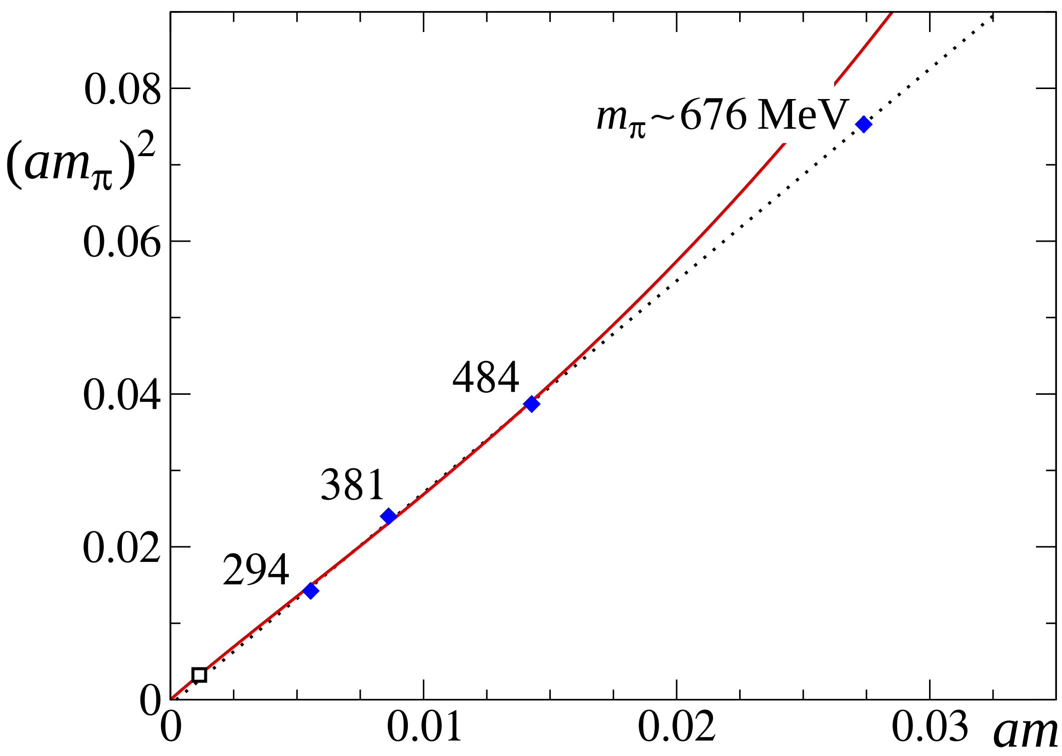
- Theoretical prediction is

  $a_0^0 - a_0^2 = 0.265 \pm 0.004$ Colangelo et al. 2001

- Here, isospin breaking plays a central role. Theoretical understanding is underdeveloped

Lattice calculations

- Lattice methods now reach the domain where it becomes possible to make contact with ChPT



Lüscher, Lattice conference 2005

No quenching, quark masses are sufficiently light ⇒ legitimate to use ChPT for the extrapolation to the physical values of m_u, m_d

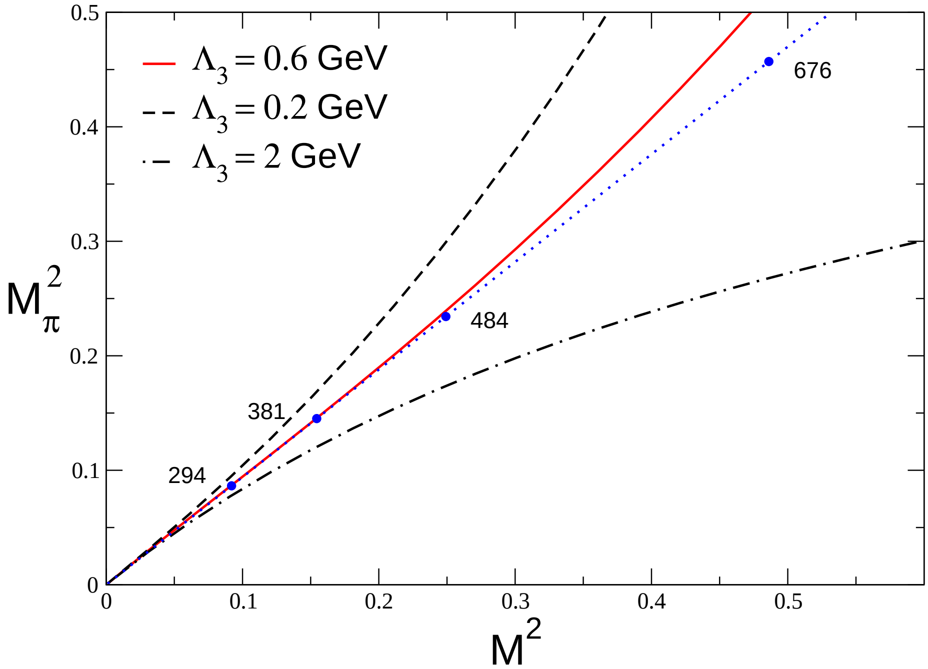
- Expansion of M_π^2 in powers of the quark mass:

$$M_\pi^2 = M^2 \left\{1 - \frac{1}{2} x \ln \frac{\Lambda_3^2}{M^2} + O(x^2)\right\}$$

$$x = \frac{M^2}{(4\pi F)^2}\,, \qquad M^2 = 2Bm$$

- Value of Λ_3 is not known well, makes the difference between ordinary and generalized ChPT
- Crude estimate based on SU(3): GL 1984

$$\bar{\ell}_3 = 2.9 \pm 2.4 \rightarrow 0.2\,\text{GeV} < \Lambda_3 < 2\,\text{GeV}$$



$\Rightarrow$ can expect better estimates for Λ_3 soon

- Further illustration: MILC data on the quark mass dependence of M_π, M_K, F_π, F_K

  Problem: butchering of fermion determinant

- Results neatly confirm the presence of chiral logarithms, data allow crude determination of L_4, L_5, L_6, L_8

- Despite the fact that m_u, m_d are tiny, the value of F_π is predicted to decrease significantly if m_u, m_d are sent to zero:

$$\frac{F_\pi}{F} = 1.072 \pm 0.004 \qquad \text{ACCGL 2004}$$

  F is value for $m_u = m_d = 0$ (m_s kept fixed)

- The MILC data pass this test remarkably well:

$$\frac{F_\pi}{F} = 1.06 \pm 0.01 \qquad \text{MILC 2004}$$

  Effect is indeed 3 or 4 times larger than what might be expected from $F_K/F_\pi \simeq 1.22$

- Phenomenon is well understood: final state interaction generates a large chiral logarithm

- Low energy theorem for scalar radius:

$$\frac{F_\pi}{F} = 1 + \frac{1}{6} M_\pi^2 \langle r^2 \rangle_s + \frac{13 M_\pi^2}{192\,\pi^2 F_\pi^2} + O(M_\pi^4)$$

⇒ Can use lattice data on the decay constants to measure the scalar radius

$$\langle r^2 \rangle_s = 0.5 \pm 0.1 \text{ fm}^2 \qquad \text{MILC 2004}$$
$$\langle r^2 \rangle_s = 0.61 \pm 0.04 \text{ fm}^2 \qquad \text{CGL 2001}$$

- Significant progress also in lattice evaluations of the exotic S-wave scattering length

$$a_0^2 = -0.0426 \pm 0.0006\text{(stat.)} \pm 0.0003\text{(syst.)} \pm 0.0018\text{(theor.)} \qquad \text{NPLQCD 2005}$$

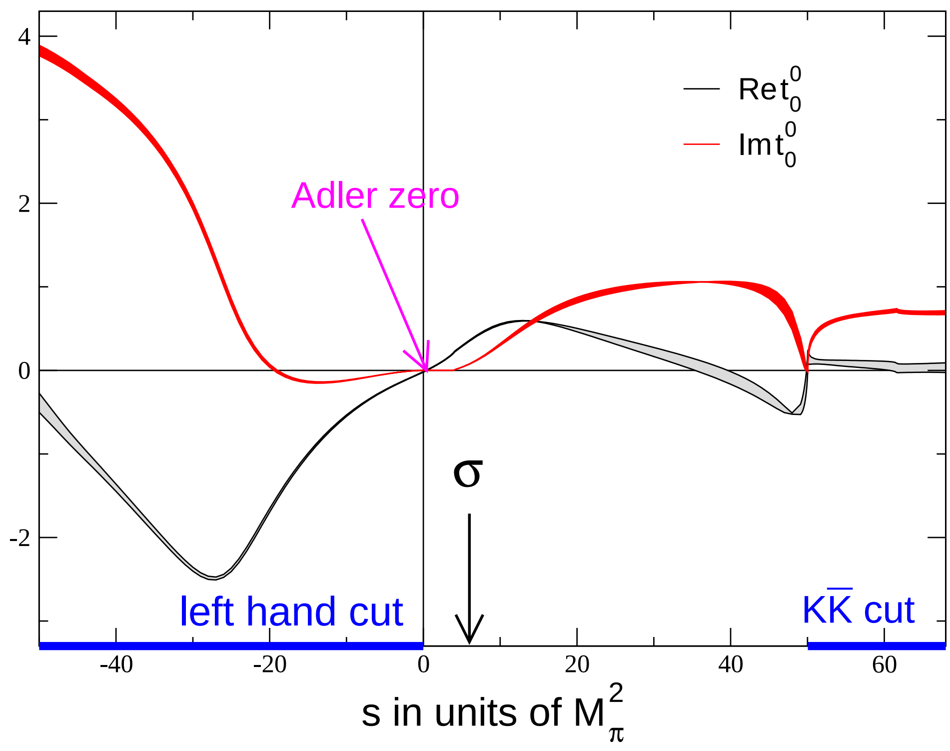
To be compared with the theoretical prediction (Weinberg 1966 + corrections to two loops)

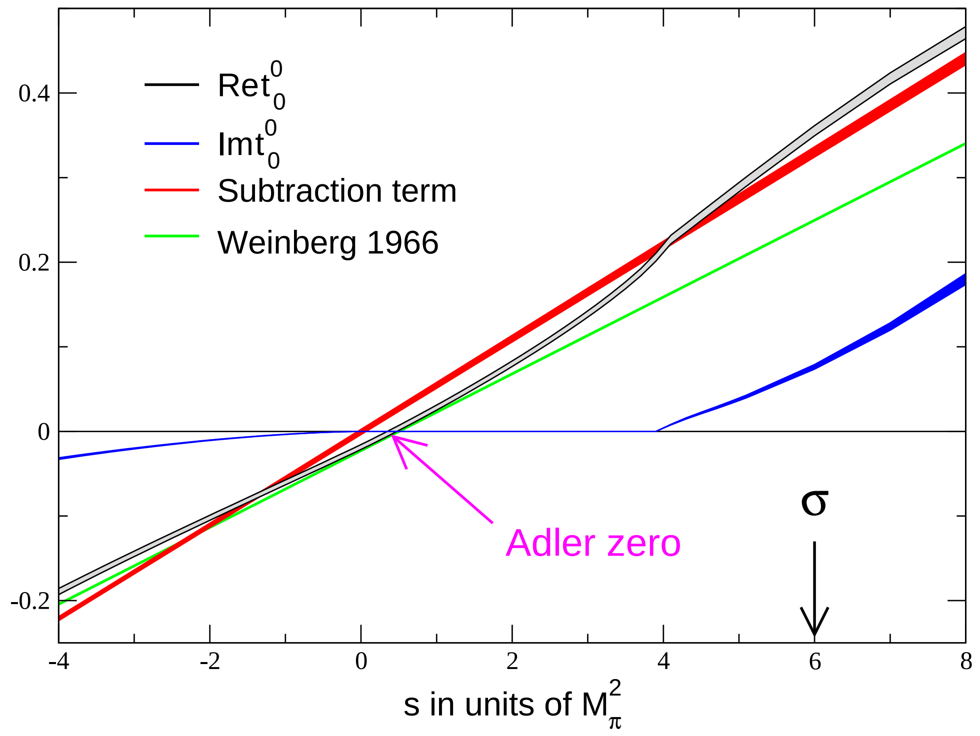
$$a_0^2 = -0.0444 \pm 0.0010 \qquad \text{CGL 2001}$$

Mass and width of the σ

D. Bugg, I. Caprini, G. Colangelo & H. L., ongoing work

- Analyticity, unitarity and crossing symmetry are essential here – extrapolation of purely phenomenological parametrizations ambiguous
- Pole occurs at low value of s, closer to left hand cut than to singularities from $K\bar{K}$, $f_0(980)$
- Crossing symmetry $\longrightarrow$ can calculate the amplitude also on the left hand cut





- Theoretical constraints are particularly strong for $0 < s < 4M_\pi^2$:
  - Low energy theorems
  - Martin inequalities
  - Sum rules of Balachandran, Nuyts & Roskies

- Do the solutions of the Roy equations have a pole in the low energy region ?
- First remove the pole from the $f_0(980)$

$$\eta_0^0 \exp 2\, i\, \delta_0^0 = S_{f_0}\, S_\sigma$$

- Use an N/D representation for S_σ

$$S_\sigma = 1 + 2\, i\, \rho\, t_\sigma, \qquad t_\sigma = \frac{N_\sigma}{D_\sigma}, \qquad \rho = \sqrt{1 - 4M_\pi^2/s}$$

- Adler zero $\Rightarrow N_\sigma = (s - s_A)$
- Unitarity in the elastic region $\Rightarrow \text{Im}\, D_\sigma = -N_\sigma\, \rho$
- Old trick due to Gounaris & Sakurai: ρ is the imaginary part of the scalar loop integral j

$\Rightarrow$ the function $h_\sigma = D_\sigma + N_\sigma\, j$ is real

$\Rightarrow$ Representation for the partial wave

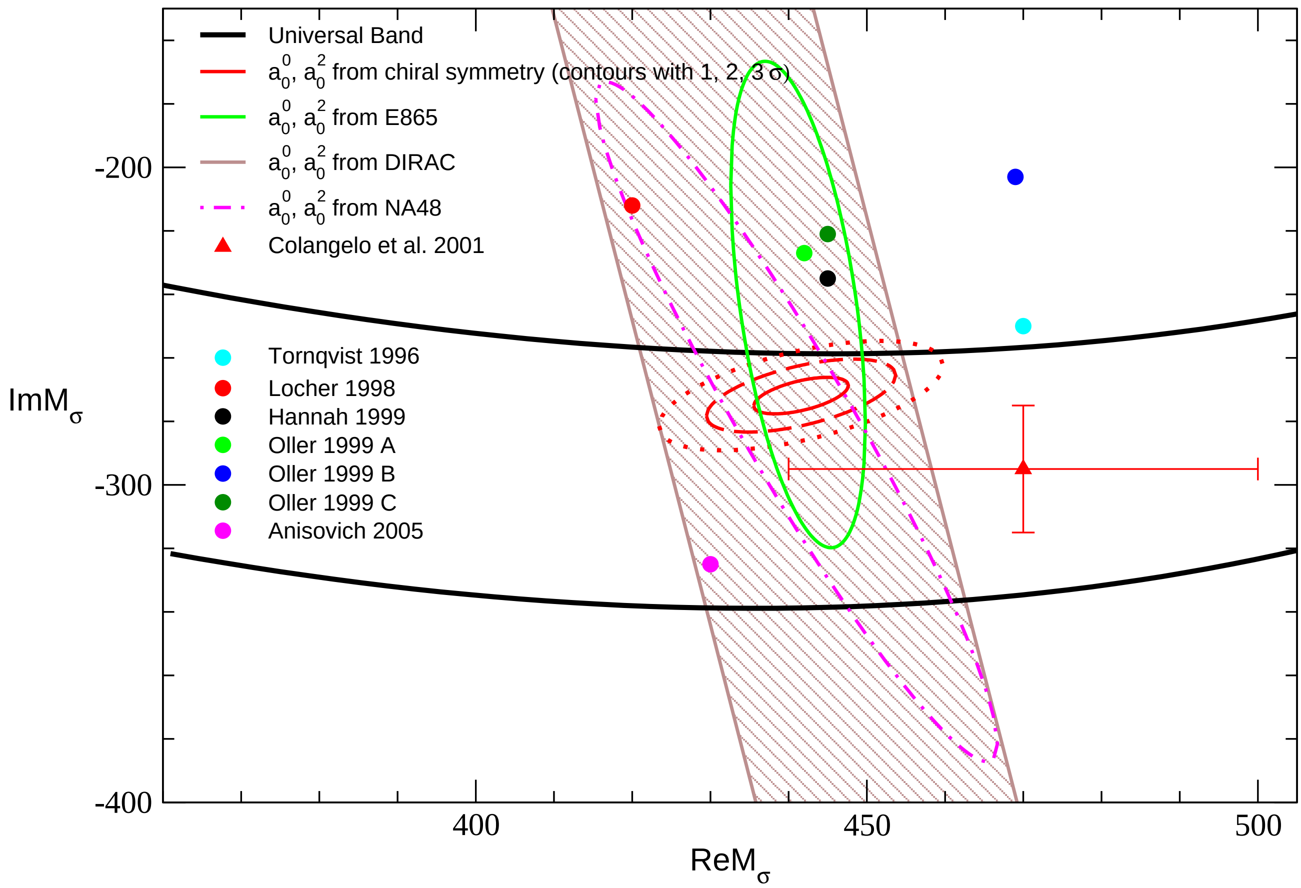
$$t_\sigma = \frac{N_\sigma}{h_\sigma - N_\sigma j}$$

h_σ does not have an elastic cut, can be continued analytically into the second sheet

⇒ Can perform the continuation with the solutions of the Roy equations

⇒ $t_\sigma(s)$ does contain a pole near threshold

- position is controlled by a_0^0, a_0^2, δ_0^0(800 MeV)
- preliminary estimate: $\sqrt{s} \simeq 442 - i\,272$ MeV



Scalar form factor

$$\langle\pi|\, m_u\bar{u}u + m_d\bar{d}d\,|\pi\rangle = \sigma_\pi\, f(t)$$
$$f(t) = 1 + \tfrac{1}{6}\,\langle r^2\rangle_s\, t + O(t^2)$$

- Value at $t = 0$: σ-term of the pion

$$\sigma_\pi = m_u \frac{\partial M_\pi^2}{\partial m_u} + m_d \frac{\partial M_\pi^2}{\partial m_d} \simeq M_\pi^2$$

- Slope at $t = 0$: *scalar radius* $\langle r^2\rangle_s$
  plays an important role in ChPT , because it determines the sensitivity of F_π to m_u, m_d

$$\frac{F_\pi}{F} = 1 + \frac{1}{6} M_\pi^2 \langle r^2\rangle_s + \frac{13 M_\pi^2}{192\,\pi^2 F_\pi^2} + O(M_\pi^4)$$

- There is an analogous formula also for F_K/F
  If Zweig rule violating contributions are dropped

$$\frac{F_K}{F_\pi} = 1 + \frac{1}{6}(M_K^2 - M_\pi^2)\langle r^2\rangle_s + \chi\text{logs}$$
$$\Rightarrow \quad \langle r^2\rangle_s = 0.55 \pm 0.15\,\text{fm}^2$$

Gasser & L. 1985

Dispersive analysis

- Early work was motivated by the search for a very light Higgs meson

Truong & Willey 1989
Donoghue, Gasser & L. 1990

- Assume that $f(t)$ does not have zeros
  $\Rightarrow$ $f(t)$ is determined by its phase $\delta_f(t)$

$$f(t) = |f(t)| e^{i\,\delta_f(t)}$$
$$f(t) = \exp \frac{t}{\pi} \int_{4M_\pi^2}^{\infty} ds \, \frac{\delta_f(s)}{s(s-t)}$$

In particular, $\langle r^2 \rangle_s$ can be calculated from $\delta_f(t)$

$$\langle r^2 \rangle_s = \frac{6}{\pi} \int_{4M_\pi^2}^{\infty} ds \, \frac{\delta_f(s)}{s^2}$$

- Watson theorem:

$$\delta_f(t) = \delta_0^0(t) \qquad t < 16\, M_\pi^2$$

- Phenomenology: inelasticity remains very small below $4\, M_K^2$ $\Rightarrow$ $\delta_f(t) \simeq \delta_0^0(t)$ holds for $t < 4 M_K^2$

$$\frac{6}{\pi} \int_{4M_\pi^2}^{4M_K^2} ds\, \frac{\delta_f(s)}{s^2} \simeq 0.42\, \mathrm{fm}^2$$

- $K\bar{K}$ channel dominates the inelasticity

$\Rightarrow$ two-channel version of the Omnes formula

$$\langle r^2 \rangle_s = 0.61 \pm 0.04\, \mathrm{fm}^2$$

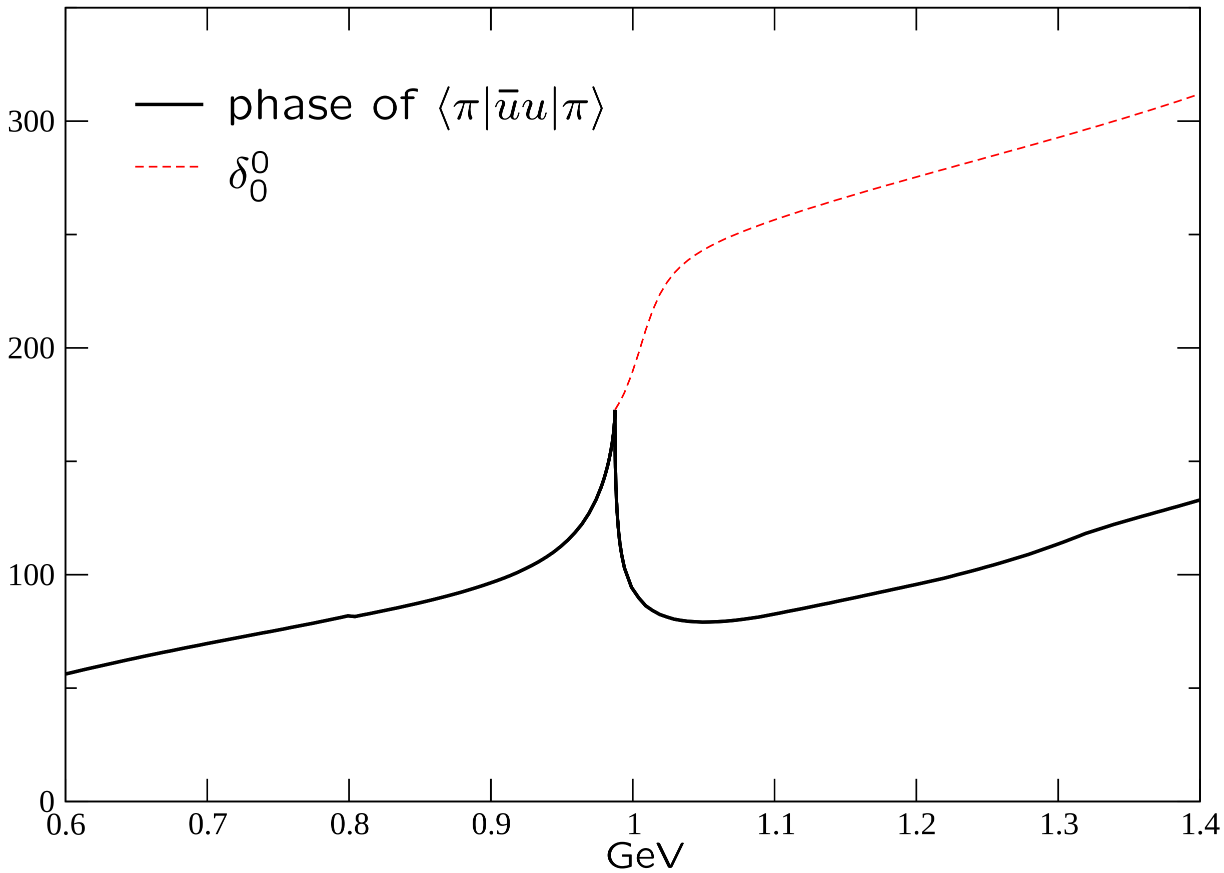
DGL 1990, CGL 2001

- Thorough analysis of the problem, including contributions from other inelastic channels:

$$\langle r^2 \rangle_s = 0.58\, \mathrm{to}\, 0.65\, \mathrm{fm}^2$$

Moussallam 1999

Phase of the scalar form factor



Ananthanarayan, Caprini, Colangelo, Gasser & L. 2004

- Behaviour near $K\bar{K}$ threshold:
  – δ_0^0 rapidly grows
  – phase of the form factor rapidly drops

Slope of the scalar $K\pi$ form factor

$$\lambda_0 = \frac{1}{6}\, M_\pi^2\, \langle r^2 \rangle_{K\pi}$$

- Can be measured in $K \to \pi\mu\nu$ decay
- Prediction based on ChPT to one loop:

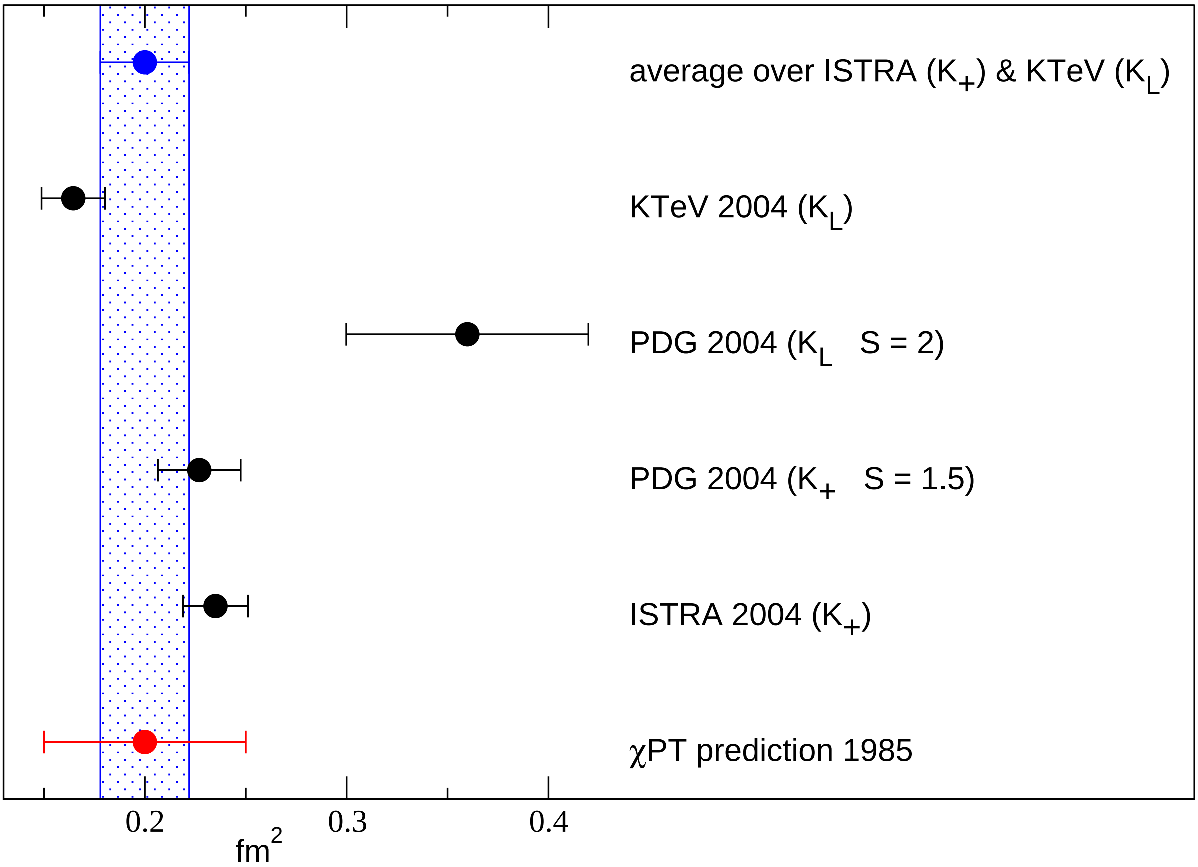
$$\langle r^2 \rangle_{K\pi} = 0.20 \pm 0.05\,\text{fm}^2$$

Gasser & L. 1985

- $\sim$ 3 times smaller than scalar radius of pion ! ChPT predicts very strong symmetry breaking in the scalar radii, weak s.b. in the vector radii
- Experimental situation was not clear in 1985

  A high statistics experiment (Donaldson 1974) was in agreement with the theoretical expectations, but more recent ones (Clark 1977, Hill 1979, Cho 1980, Birulev 1981) were in flat contradiction with chiral symmetry.
- During the last year, the experimental situation improved very significantly

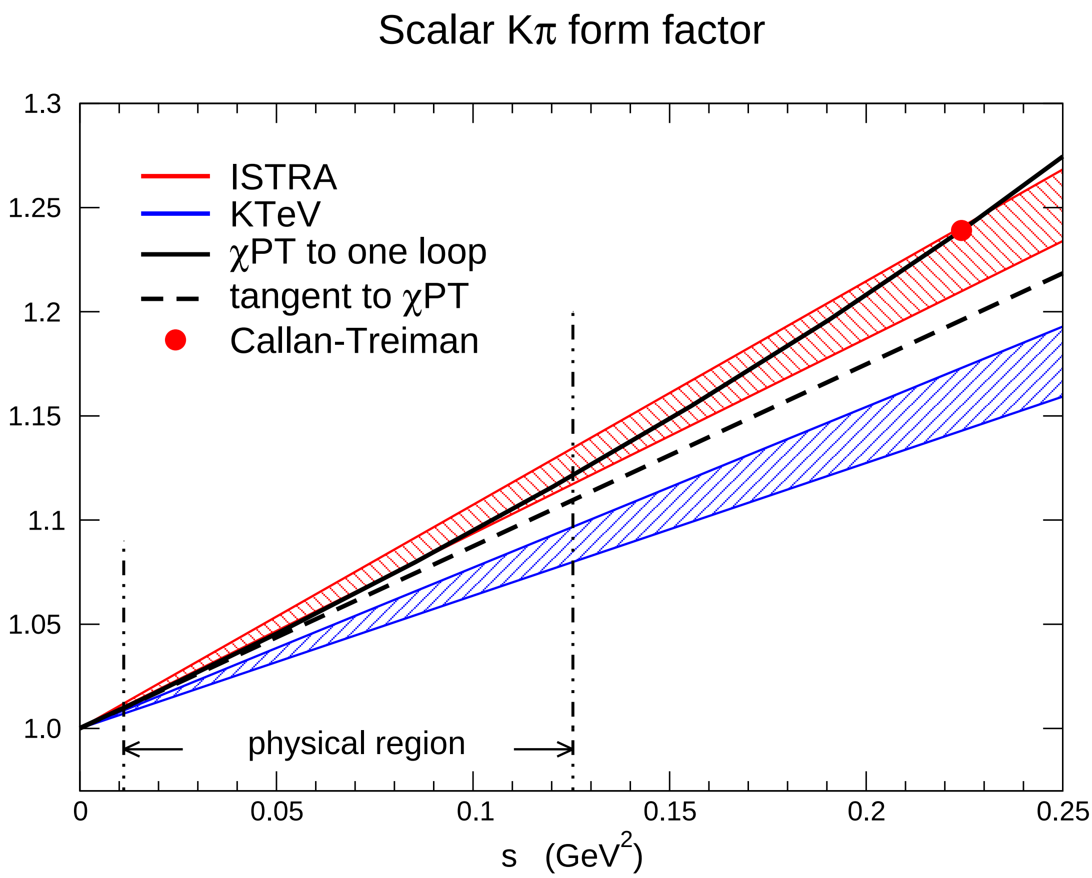
Mean square radius of the scalar Kπ form factor



$$\text{ISTRA:} \quad \langle r^2\rangle_{K\pi} = 0.235 \pm 0.014 \pm 0.008 \text{ fm}^2$$

$$\text{KTeV:} \quad \langle r^2\rangle_{K\pi} = 0.165 \pm 0.016 \text{ fm}^2$$

$$\text{ChPT :} \quad \langle r^2\rangle_{K\pi} = 0.20 \pm 0.05 \,\text{fm}^2$$



Plot shows normalized form factor $f_0(s)/f_0(0)$

- Callan-Treiman-relation:

$$f_0(M_K^2 - M_\pi^2) = \frac{F_K}{F_\pi} + O(m_u, m_d)$$

Correction is tiny: no term of order $M_\pi^2 \log M_\pi^2$
(leading order in SU(3)×SU(3) : 3 permille)

- Curvature not negligible at this precision,
is due to $I = \frac{1}{2}$ $K\pi$ final state interaction

Progress on the theoretical side

- Form factors are now known to two loops

Post & Schilcher, Bijnens & Talavera

- Extension of Roy analysis to $K\pi$ scattering: Roy-Steiner equations, yield reliable results for the behaviour of the phases below 1 GeV

  Estimate for the subtraction constants on the basis of the available data

  $a_0^{1/2} = 0.224 \pm 0.022 \qquad a_0^{3/2} = 0.0448 \pm 0.0077$

Büttiker, Descotes-Genon & Moussallam

⇒ The Roy-Steiner equations can now be matched with the two loop representation of ChPT

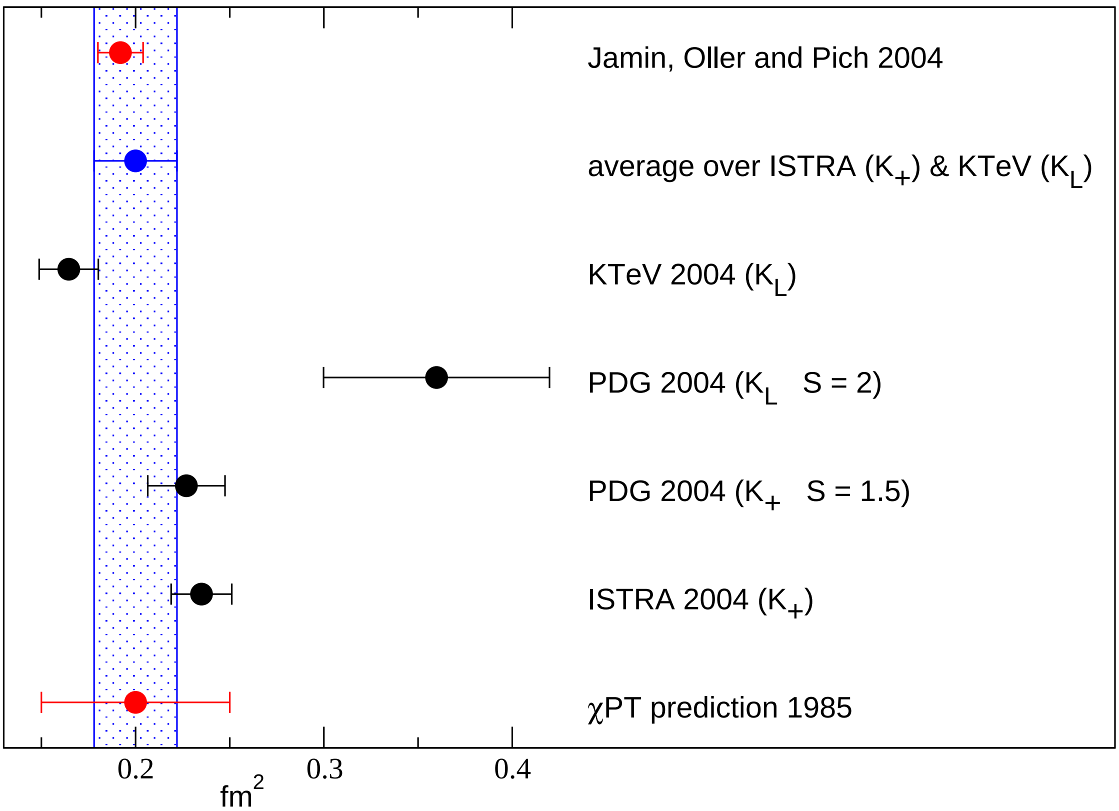
⇒ On this basis, the low energy theorems of SU(2)×SU(2) and SU(3)×SU(3) can then be analyzed in a controlled manner

- Dispersive analysis of the $K\pi$ form factors allows to determine their curvature

  The Callan-Treiman relation then leads to a much sharper prediction for the radius:

$$\langle r^2 \rangle_{K\pi} = 0.192 \pm 0.012 \text{ fm}^2$$

Jamin, Oller and Pich 2004



⇒ In view of the improved experimental situation, expect significant progress in kaon physics soon