The lowest resonance of QCD

H. Leutwyler

University of Bern

Minneapolis, May 11, 2006

$\pi\pi$ interaction

- Plays a crucial role whenever the strong interaction is involved at low energies

  Example: Standard model prediction for muon magnetic moment

- Main experiments on $\pi\pi$ scattering were done in the seventies. What's new ?
  - Significant theoretical progress, based on ChPT + dispersion theory
  - New precision data:

$K \to \pi\pi\ell\nu$	E865	Brookhaven
pionic atoms	DIRAC	CERN
$K \to 3\pi$	NA48/2	CERN

  - Lattice results on M_π, F_π, a_0^2, $\langle r^2 \rangle_s$

Analyticity and crossing

- $\pi\pi$ scattering is special: crossed channels are identical

$\Rightarrow$ Re $T(s,t)$ can be represented as a twice subtracted dispersion integral over Im $T(s,t)$ in physical region

S.M. Roy 1971

- The 2 subtraction constants can be identified with the S-wave scattering lengths:

$$a_0^0\ ,\ a_0^2 \begin{array}{l}\leftarrow \text{isospin}\\ \leftarrow \text{angular momentum}\end{array}$$

- Representation leads to dispersion relations for the individual partial waves: *Roy equations*

Roy equations

- Pioneering work on the physics of the Roy equations: Basdevant, Froggatt & Petersen 1974
- Dispersion integrals converge rapidly (2 subtractions)

$\Rightarrow$ Crude phenomenological information on Im $T(s,t)$ for energies above 800 MeV suffices

$\Rightarrow$ Given a_0^0, a_0^2, the scattering amplitude can be calculated to within small uncertainties

Ananthanarayan, Colangelo, Gasser & L. 2001
Descotes, Fuchs, Girlanda & Stern 2002

$\Rightarrow$ a_0^0, a_0^2 are the essential parameters at low energy

- Main problem in early work: a_0^0, a_0^2 poorly known
Experimental information near threshold is meagre

Low energy theorems

- Chiral perturbation theory provides the missing piece: theoretical prediction for a_0^0, a_0^2

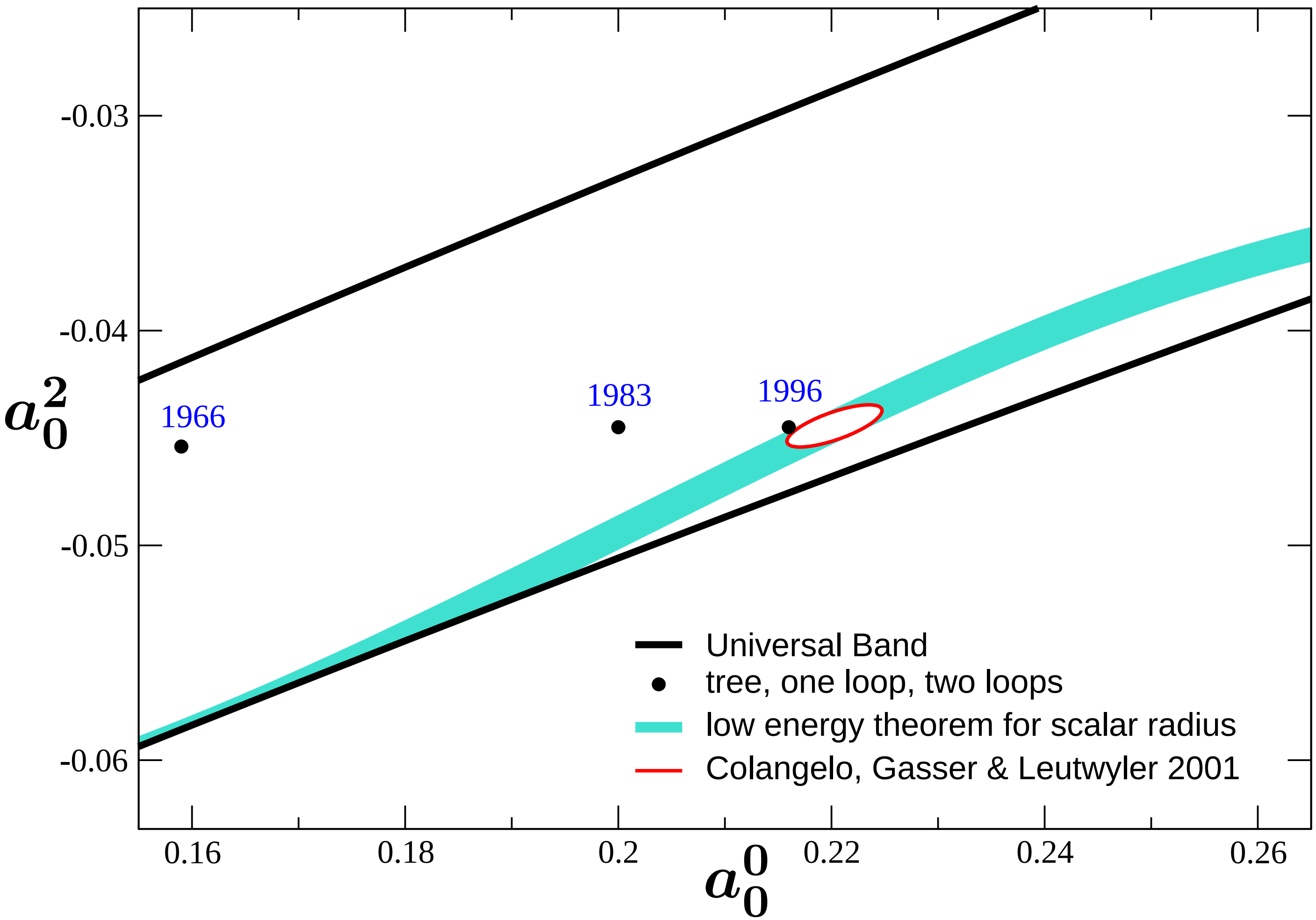
  Weinberg 1966, Gasser & L. 1983, Bijnens, Colangelo, Ecker, Gasser & Sainio 1996

- Most accurate results for a_0^0, a_0^2 are obtained by matching the chiral and dispersive representations near the center of the Mandelstam triangle

  Colangelo, Gasser & L. 2001

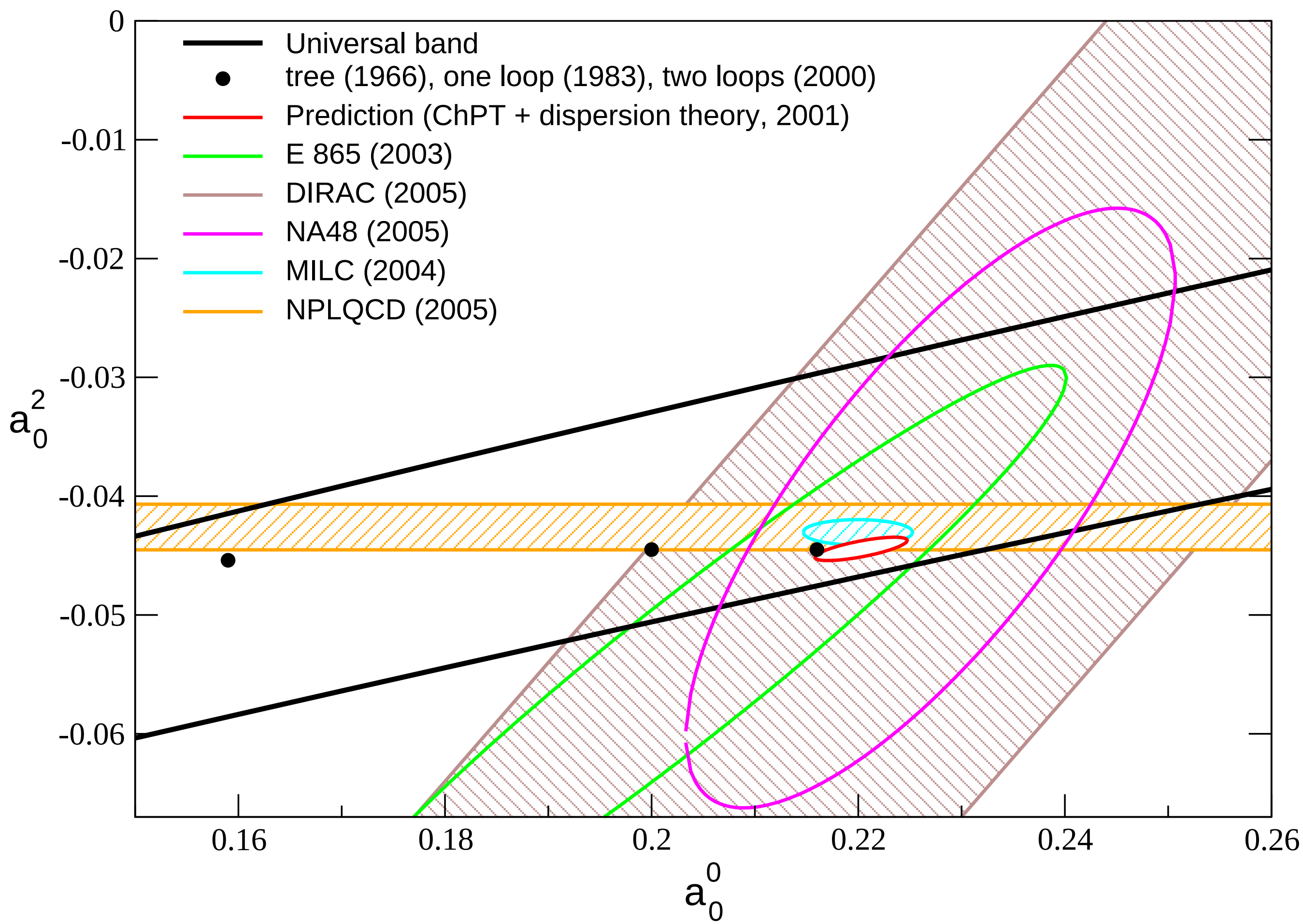
- In combination with the low energy theorems for a_0^0, a_0^2, the dispersion relations for the partial waves fix the $\pi\pi$ scattering amplitude to an incredible degree of accuracy

Predictions for the S-wave $\pi\pi$ scattering lengths



Sizeable corrections in a_0^0, while a_0^2 nearly stays put

Tests of the predictions for a_0^0, a_0^2: experiment & lattice



Theory is ahead of experiment . . .

The σ

I. Caprini, G. Colangelo and H. Leutwyler, Phys. Rev. Lett. 96 (2006) 132001

- Does QCD have a resonance near threshold ?
- Why care ?
  - Concerns the nonperturbative domain of QCD
  - Quark and gluon degrees of freedom useless there

  $\Rightarrow$ Understanding very poor, pattern of energy levels ?
  - Lowest resonance: σ ? ρ ?
- Resonance $\leftrightarrow$ pole on second sheet
  - Poles are universal
  - Pole position is unambiguous, even if width is large
  - Where is the pole closest to the origin ?

$f_0(600)$ or σ

$I^G(J^{PC}) = 0^+(0^{++})$

A REVIEW GOES HERE – Check our WWW List of Reviews

$f_0(600)$ T-MATRIX POLE $\sqrt{s}$

Note that $\Gamma \approx 2\ \mathrm{Im}(\sqrt{s_{\mathrm{pole}}})$.

VALUE (MeV)	*DOCUMENT ID*		*TECN*	*COMMENT*
(400–1200)−i(300–500) OUR ESTIMATE				
• • • We do not use the following data for averages, fits, limits, etc. • • •				
$(541 \pm 39)-i(252 \pm 42)$	[1] ABLIKIM	04A	BES2	$J/\psi \to \omega\pi^+\pi^-$
$(528 \pm 32)-i(207 \pm 23)$	[2] GALLEGOS	04	RVUE	Compilation
$(440 \pm 8)-i(212 \pm 15)$	[3] PELAEZ	04A	RVUE	$\pi\pi \to \pi\pi$
$(533 \pm 25)-i(247 \pm 25)$	[4] BUGG	03	RVUE	
$532 - i272$	BLACK	01	RVUE	$\pi^0\pi^0 \to \pi^0\pi^0$
$(470 \pm 30)-i(295 \pm 20)$	[5] COLANGELO	01	RVUE	$\pi\pi \to \pi\pi$
$(535^{+48}_{-36})-i(155^{+76}_{-53})$	[6] ISHIDA	01		$\Upsilon(3S) \to \Upsilon\pi\pi$
$610 \pm 14 - i620 \pm 26$	[7] SUROVTSEV	01	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$(558^{+34}_{-27})-i(196^{+32}_{-41})$	ISHIDA	00B		$p\overline{p} \to \pi^0\pi^0\pi^0$
$445 - i235$	HANNAH	99	RVUE	π scalar form factor
$(523 \pm 12)-i(259 \pm 7)$	KAMINSKI	99	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$, $\sigma\sigma$
$442 - i\ 227$	OLLER	99	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$469 - i203$	OLLER	99B	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$445 - i221$	OLLER	99C	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$, $\eta\eta$
$(1530^{+\ 90}_{-250})-i(560 \pm 40)$	ANISOVICH	98B	RVUE	Compilation
$420 - i\ 212$	LOCHER	98	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$(602 \pm 26)-i(196 \pm 27)$	[8] ISHIDA	97		$\pi\pi \to \pi\pi$
$(537 \pm 20)-i(250 \pm 17)$	[9] KAMINSKI	97B	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$, 4π
$470 - i250$	[10,11] TORNQVIST	96	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$, $K\pi$, $\eta\pi$
$\sim (1100 - i300)$	AMSLER	95B	CBAR	$\overline{p}p \to 3\pi^0$
$400 - i500$	[11,12] AMSLER	95D	CBAR	$\overline{p}p \to 3\pi^0$
$1100 - i137$	[11,13] AMSLER	95D	CBAR	$\overline{p}p \to 3\pi^0$
$387 - i305$	[11,14] JANSSEN	95	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$525 - i269$	[15] ACHASOV	94	RVUE	$\pi\pi \to \pi\pi$
$(506 \pm 10)-i(247 \pm 3)$	KAMINSKI	94	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$370 - i356$	[16] ZOU	94B	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$408 - i342$	[11,16] ZOU	93	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$870 - i370$	[11,17] AU	87	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$470 - i208$	[18] BEVEREN	86	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$, $\eta\eta$, ...
$(750 \pm 50)-i(450 \pm 50)$	[19] ESTABROOKS	79	RVUE	$\pi\pi \to \pi\pi$, $K\overline{K}$
$(660 \pm 100)-i(320 \pm 70)$	PROTOPOP...	73	HBC	$\pi\pi \to \pi\pi$, $K\overline{K}$
$650 - i370$	[20] BASDEVANT	72	RVUE	$\pi\pi \to \pi\pi$

Model independent determination of the pole

- All of the results quoted by the PDG are obtained by
  (a) parametrizing the data for real values of s
  (b) continuing this parametrization analytically in s

  $\Rightarrow$ Result is sensitive to the parametrization used

Model independent determination of the pole

- All of the results quoted by the PDG are obtained by
  (a) parametrizing the data for real values of s
  (b) continuing this parametrization analytically in s

  $\Rightarrow$ Result is sensitive to the parametrization used

- We found a model independent method:
  1. Poles on second sheet are zeros on first sheet
  2. The Roy equations are valid for complex values of s, in a limited region of the first sheet

  $\Rightarrow$ Exact representation of the partial waves in terms of observable quantities, valid for complex values of s

Model independent determination of the pole

- All of the results quoted by the PDG are obtained by
  (a) parametrizing the data for real values of s
  (b) continuing this parametrization analytically in s

  $\Rightarrow$ Result is sensitive to the parametrization used

- We found a model independent method:

  1. Poles on second sheet are zeros on first sheet
  2. The Roy equations are valid for complex values of s, in a limited region of the first sheet

  $\Rightarrow$ Exact representation of the partial waves in terms of observable quantities, valid for complex values of s

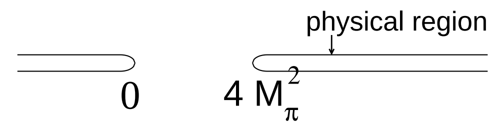
  3. Can evaluate this representation to good precision and determine the zeros numerically

Pole on second sheet $\longleftrightarrow$ zero on first sheet

- $S^0_0(s) = \eta^0_0(s)\exp 2i\delta^0_0(s)$

  $S^0_0(s)$ is analytic in the cut plane

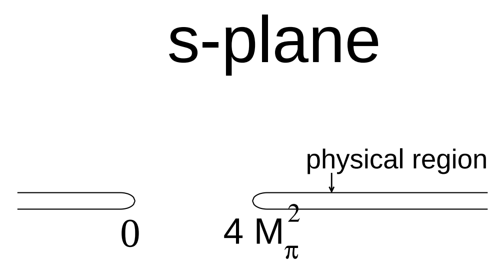
s-plane



Pole on second sheet $\leftrightarrow$ zero on first sheet

- $S_0^0(s) = \eta_0^0(s) \exp 2i\delta_0^0(s)$

  $S_0^0(s)$ is analytic in the cut plane

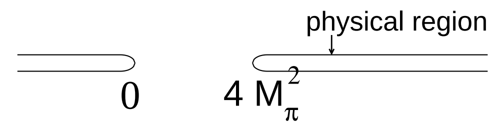


- For $0 < s < 4M_\pi^2$, $S_0^0(s)$ is real

$\Rightarrow$ $S_0^0(s^\star) = S_0^0(s)^\star$

x in elastic interval: $S_0^0(x \pm i\epsilon) = \exp \pm 2i\delta_0^0(x)$

Pole on second sheet $\leftrightarrow$ zero on first sheet



- $S_0^0(s) = \eta_0^0(s)\exp 2i\delta_0^0(s)$

  $S_0^0(s)$ is analytic in the cut plane

- For $0 < s < 4M_\pi^2$, $S_0^0(s)$ is real

$\Rightarrow S_0^0(s^\star) = S_0^0(s)^\star$

x in elastic interval: $S_0^0(x \pm i\epsilon) = \exp \pm 2i\delta_0^0(x)$

- Second sheet is reached by continuation across the elastic interval of the right hand cut

$$S_0^0(x - i\epsilon)^{II} = S_0^0(x + i\epsilon)^I = 1/S_0^0(x - i\epsilon)^I$$

Analyticity $\Rightarrow$ $\boxed{S_0^0(s)^{II} = 1/S_0^0(s)^I}$ valid $\forall s$

Pole in $S_0^0(s)^{II} \Longleftrightarrow$ zero in $S_0^0(s)^I$

Roy equation for the isoscalar S-wave

$$S^0_0(s) = 1 + 2\,i\rho\,t^0_0(s) \qquad\qquad \rho = \sqrt{1 - 4M_\pi^2/s}$$

Roy equation for the isoscalar S-wave

$$S_0^0(s) = 1 + 2\, i \rho\, t_0^0(s) \qquad \rho = \sqrt{1 - 4M_\pi^2/s}$$

$$t_0^0(s) = a + (s - 4M_\pi^2)\, b + \int_{4M_\pi^2}^{\infty} ds' \left\{ K_0(s,s')\, \mathrm{Im}\, t_0^0(s') \right.$$

$$\left. + K_1(s,s')\, \mathrm{Im}\, t_1^1(s') + K_2(s,s')\, \mathrm{Im}\, t_0^2(s') \right\}$$

$+$ higher partial waves

Roy equation for the isoscalar S-wave

$$S_0^0(s) = 1 + 2\,i\rho\, t_0^0(s) \qquad\qquad \rho = \sqrt{1 - 4M_\pi^2/s}$$

$$t_0^0(s) = a + (s - 4M_\pi^2)\, b + \int_{4M_\pi^2}^{\infty} ds' \,\{K_0(s,s')\,\mathrm{Im}\, t_0^0(s')$$
$$+ K_1(s,s')\,\mathrm{Im}\, t_1^1(s') + K_2(s,s')\,\mathrm{Im}\, t_0^2(s')\}$$
$$+ \text{ higher partial waves}$$

- The subtraction constants are determined by a_0^0, a_0^2:
  $a = a_0^0\,, \qquad b = (2a_0^0 - 5a_0^2)/(12M_\pi^2)$

Roy equation for the isoscalar S-wave

$$S^0_0(s) = 1 + 2\,i\rho\,t^0_0(s) \qquad\qquad \rho = \sqrt{1 - 4M_\pi^2/s}$$

$$t^0_0(s) = a + (s - 4M_\pi^2)\,b + \int_{4M_\pi^2}^{\infty} ds'\,\{K_0(s,s')\,\mathrm{Im}\,t^0_0(s')$$

$$+ K_1(s,s')\,\mathrm{Im}\,t^1_1(s') + K_2(s,s')\,\mathrm{Im}\,t^2_0(s')\}$$

$+$ higher partial waves

- The subtraction constants are determined by a^0_0, a^2_0:
  $$a = a^0_0\,, \qquad b = (2a^0_0 - 5a^2_0)/(12M_\pi^2)$$
- The kernels are elementary functions, e.g.
  $$K_0(s,s') = \underbrace{\frac{1}{\pi(s'-s)}}_{r.h.cut} + \underbrace{\frac{2\,ln\{(s+s'-4M_\pi^2)/s'\}}{3\pi(s-4M_\pi^2)} - \frac{5s'+2s-16M_\pi^2}{3\pi s'(s'-4M_\pi^2)}}_{l.h.cut}$$

Roy equation for the isoscalar S-wave

$$S_0^0(s) = 1 + 2\,i\rho\,t_0^0(s) \qquad\qquad \rho = \sqrt{1 - 4M_\pi^2/s}$$

$$t_0^0(s) = a + (s - 4M_\pi^2)\,b + \int_{4M_\pi^2}^{\infty} ds'\,\{K_0(s,s')\,\mathrm{Im}\,t_0^0(s')$$

$$+\,K_1(s,s')\,\mathrm{Im}\,t_1^1(s') + K_2(s,s')\,\mathrm{Im}\,t_0^2(s')\}$$

$+$ higher partial waves

- The subtraction constants are determined by a_0^0, a_0^2:
  $a = a_0^0\,, \qquad b = (2a_0^0 - 5a_0^2)/(12M_\pi^2)$
- The kernels are elementary functions, e.g.
  $$K_0(s,s') = \underbrace{\frac{1}{\pi(s'-s)}}_{r.h.cut} + \underbrace{\frac{2\,ln\{(s+s'-4M_\pi^2)/s'\}}{3\pi(s-4M_\pi^2)} - \frac{5s'+2s-16M_\pi^2}{3\pi s'(s'-4M_\pi^2)}}_{l.h.cut}$$
- Left hand cut is essential for convergence:
  $K_0(s,s') \sim 1/s'^3$ for large s'

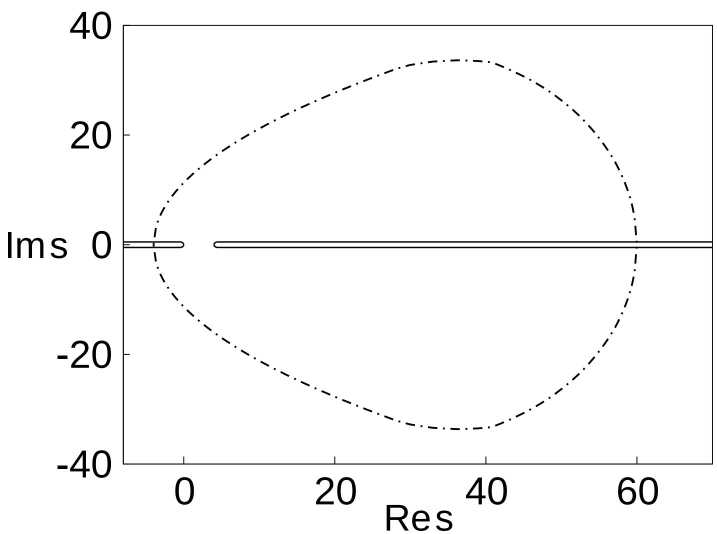
Domain of validity of the Roy equations

- Roy derived his equations for real energies in the interval $-4M_\pi^2 < s < 60M_\pi^2$

Domain of validity of the Roy equations

- Roy derived his equations for real energies in the interval $-4M_\pi^2 < s < 60M_\pi^2$
- Equations are valid for complex s in a limited region of the first sheet

I. Caprini, G. Colangelo and H. Leutwyler, Phys. Rev. Lett. 96 (2006) 132001

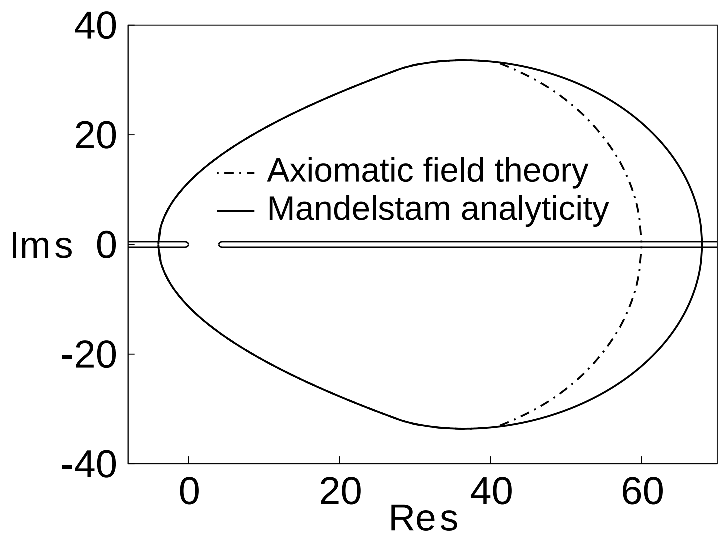


s in units of M_π^2

Domain of validity of the Roy equations

- Roy derived his equations for real energies in the interval $-4M_\pi^2 < s < 60M_\pi^2$
- Equations are valid for complex s in a limited region of the first sheet

I. Caprini, G. Colangelo and H. Leutwyler, Phys. Rev. Lett. 96 (2006) 132001



s in units of M_π^2

- Proof is based on first principles, general quantum field theory

A. Martin, *Scattering Theory: Unitarity, Analyticity and Crossing*, Lecture Notes in Physics, vol. 3, 1969.

G. Mahoux, S. M. Roy and G. Wanders, Nucl. Phys. B 70 (1974) 297.

$\Rightarrow$ Exact representation for $S_0^0(s)$ in this region
Do not need to parametrize the amplitude

Evaluation of the pole position

- Insert our solutions of the Roy equations
  For the central solution, $S_0^0(s)$ has two pairs of zeros in the region of validity of the representation:

  $$s = (6.2 \pm i\,12.3)\,M_\pi^2 \quad \sigma$$

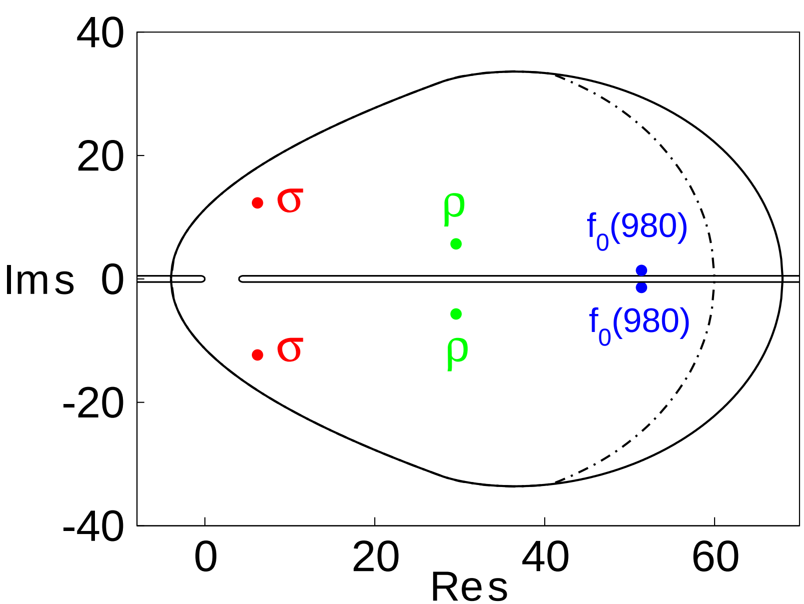
  $$s = (51.4 \pm i\,1.4)\,M_\pi^2 \quad f_0(980)$$

Evaluation of the pole position

- Insert our solutions of the Roy equations
  For the central solution, $S_0^0(s)$ has two pairs of zeros in the region of validity of the representation:

$$s = (6.2 \pm i\,12.3)\,M_\pi^2 \quad \sigma$$

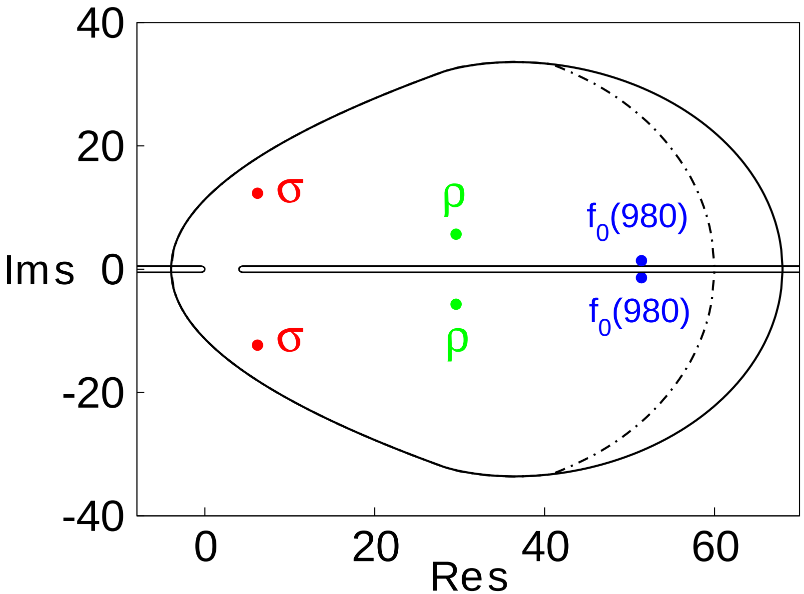
$$s = (51.4 \pm i\,1.4)\,M_\pi^2 \quad f_0(980)$$



Evaluation of the pole position

- Insert our solutions of the Roy equations
  For the central solution, $S_0^0(s)$ has two pairs of zeros in the region of validity of the representation:

$$s = (6.2 \pm i\,12.3)\,M_\pi^2 \quad \sigma$$
$$s = (51.4 \pm i\,1.4)\,M_\pi^2 \quad f_0(980)$$



s in units of M_π^2

$\Rightarrow$ 1. Lowest resonance of QCD has vacuum quantum numbers

2. Pole on lower half of sheet II occurs in vicinity of

$$m_\sigma = 441 - i\,272\,\text{MeV}$$
$$= M_\sigma - \tfrac{i}{2}\Gamma_\sigma$$

Error analysis

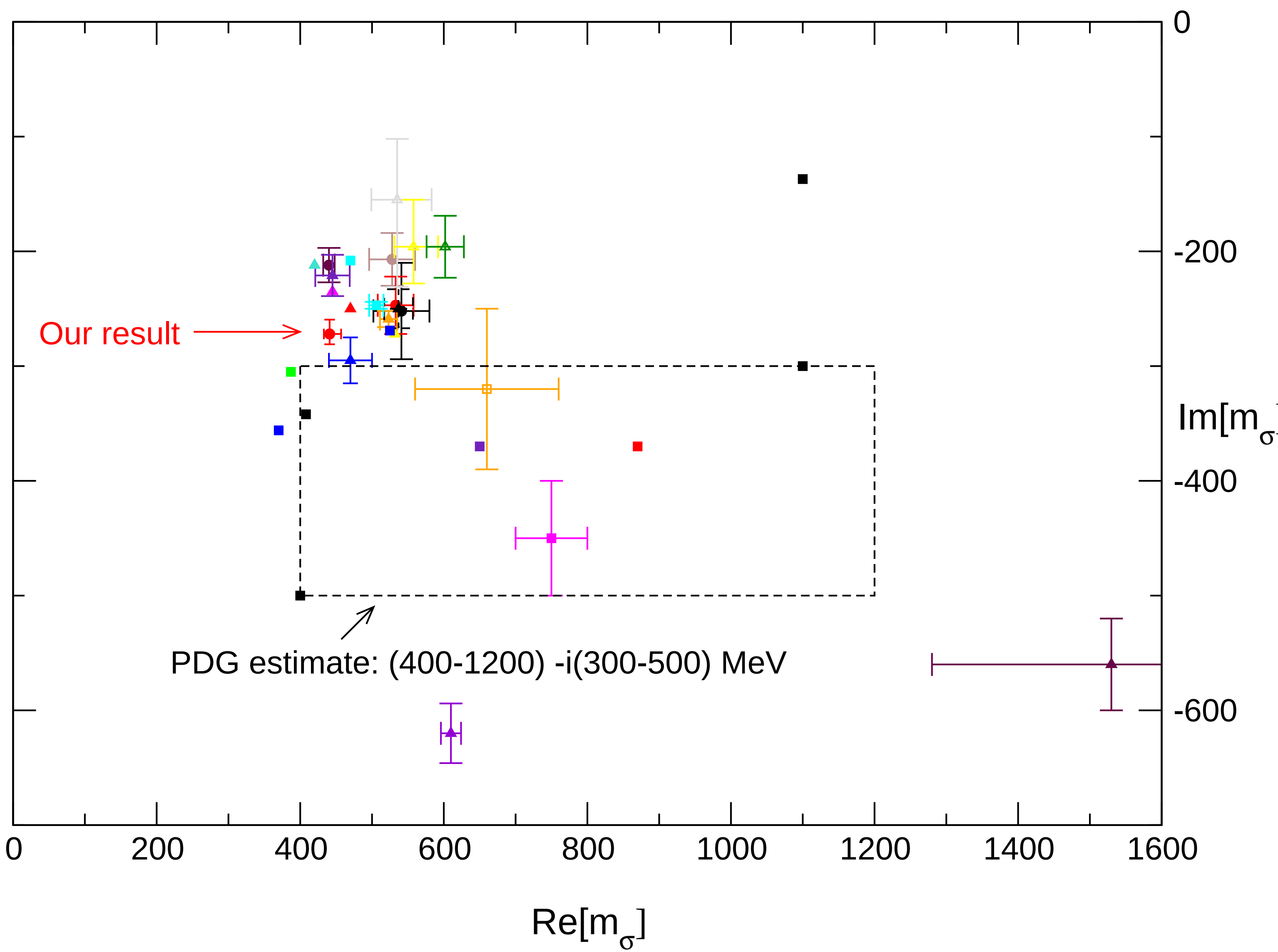
- Results depend on phenomenological input used when solving the Roy equations, subject to uncertainties
  Can follow error propagation explicitly
- Pole position of σ mainly depends on 3 input variables:

  $a_0^0, a_0^2, \delta_A \equiv \delta_0^0$(800 MeV)

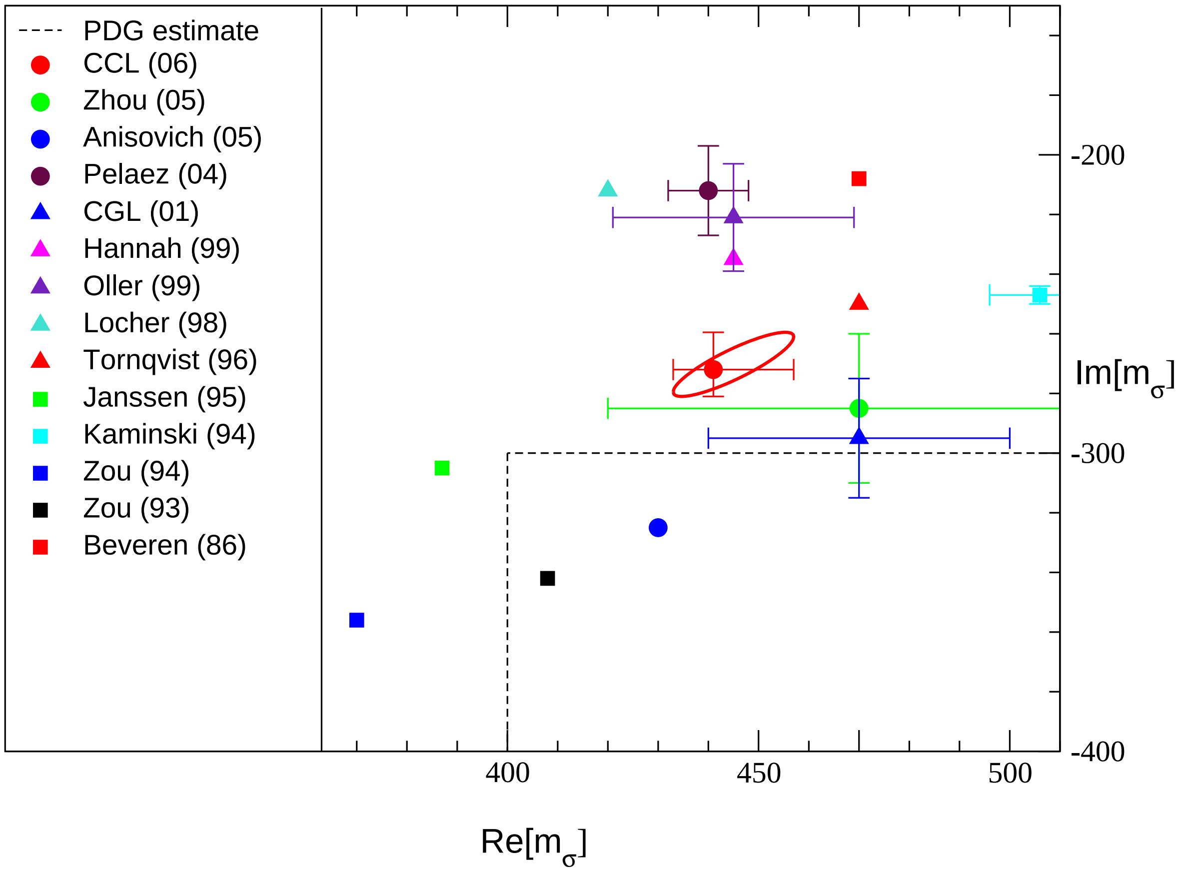
  - Substantial uncertainties in phenomenology of δ_A
  - Use the range: $\delta_A = 82.3^\circ {}^{+10^\circ}_{-4^\circ}$
- Final result, all uncertainties added up in square

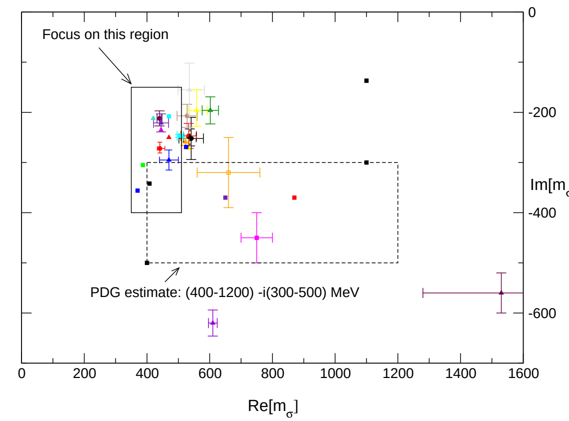
$$m_\sigma = 441\,{}^{+16}_{-8} \;-\; i\,272\,{}^{+9}_{-13}\ \text{MeV}$$

Comparison with compilation of PDG



Vicinity of the pole

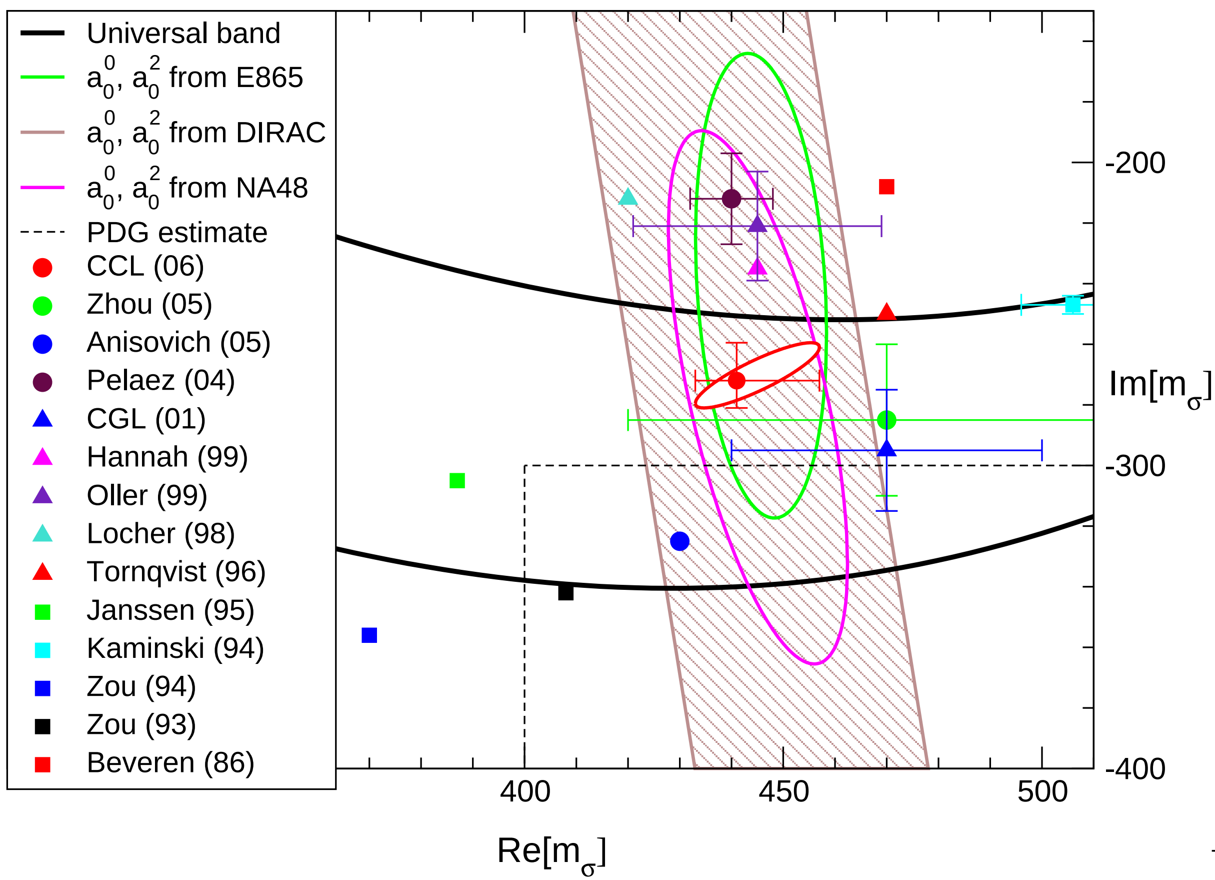




Results for Re$[m_\sigma]$ and Im$[m_\sigma]$ are strongly correlated

Ignore the theoretical predictions for a_0^0, a_0^2

- Replace the low energy theorems for a_0^0, a_0^2 by the experimental results from E865, DIRAC and NA48
- $a_0^0, a_0^2 \in$ universal band



Why are our errors so incredibly small ?

- The σ occurs at low energies
- At low energies, the subtraction term dominates

$$t_0^0(s) \simeq a_0^0 + (2a_0^0 - 5a_0^2)\frac{(s - 4M_\pi^2)}{12M_\pi^2}$$

Why are our errors so incredibly small ?

- The σ occurs at low energies
- At low energies, the subtraction term dominates

$$t_0^0(s) \simeq a_0^0 + (2a_0^0 - 5a_0^2)\frac{(s-4M_\pi^2)}{12M_\pi^2}$$

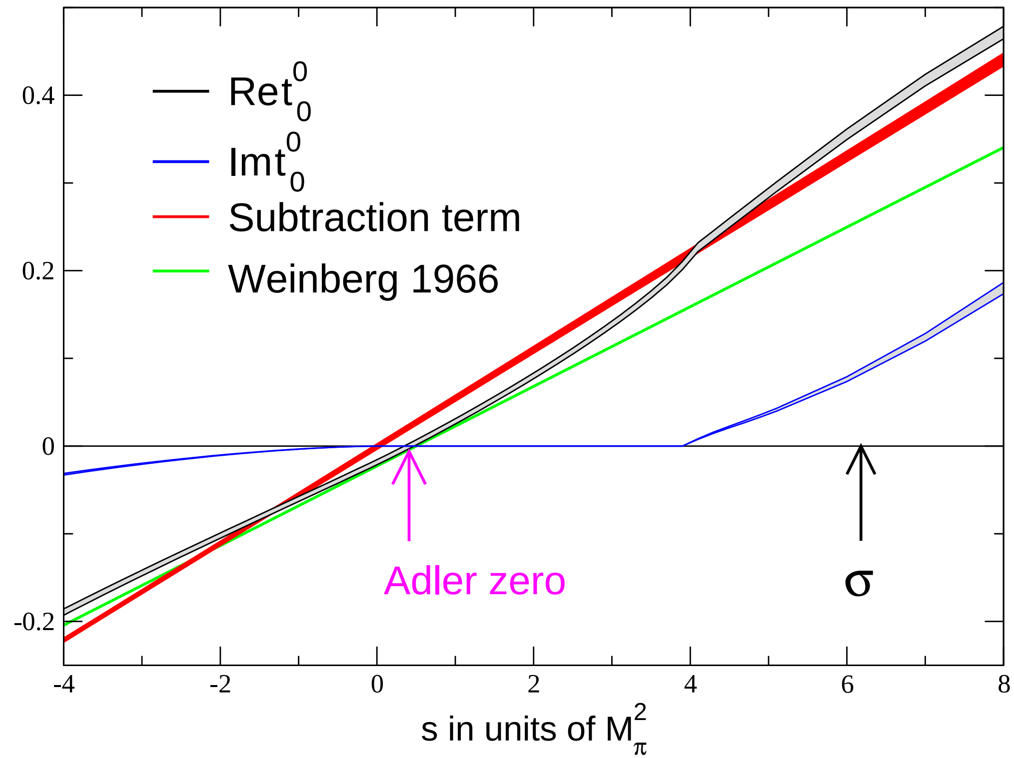
Insert low energy theorem for a_0^0, a_0^2

$\Rightarrow$ Roy equation reduces to Weinberg formula

$$t_0^0(s) \simeq \frac{(2s-M_\pi^2)}{32\pi F_\pi^2}$$

Dispersion integrals only represent a correction

At low energies, the subtraction term dominates



$$s = (0.41 \pm 0.06)\, M_\pi^2 \quad \text{Adler zero}$$

$$s = (6.2 - i\, 12.3)\, M_\pi^2 \quad \text{pole from } \sigma$$

Estimate pole position on back of an envelope

- Approximate $t_0^0(s)$ with the Weinberg formula

$$t_0^0(s) = \frac{(2s - M_\pi^2)}{32\pi F_\pi^2}$$

Where are the zeros of $S_0^0(s)$ in this approximation ?

Estimate pole position on back of an envelope

- Approximate $t_0^0(s)$ with the Weinberg formula

$$t_0^0(s) = \frac{(2s - M_\pi^2)}{32\pi F_\pi^2}$$

Where are the zeros of $S_0^0(s)$ in this approximation ?

$$1 + 2\,i\,\sqrt{1 - 4M_\pi^2/s}\;t_0^0(s) = 0$$

Estimate pole position on back of an envelope

- Approximate $t_0^0(s)$ with the Weinberg formula

$$t_0^0(s) = \frac{(2s - M_\pi^2)}{32\pi F_\pi^2}$$

Where are the zeros of $S_0^0(s)$ in this approximation ?

$$1 + 2\,i\,\sqrt{1 - 4M_\pi^2/s}\; t_0^0(s) = 0$$

$\Rightarrow$ Cubic equation for s

  - Pair of complex zeros, $m_\sigma = 365 - i\,291$ MeV

Estimate pole position on back of an envelope

- Approximate $t_0^0(s)$ with the Weinberg formula

$$t_0^0(s) = \frac{(2s - M_\pi^2)}{32\pi F_\pi^2}$$

Where are the zeros of $S_0^0(s)$ in this approximation ?

$$1 + 2\,i\,\sqrt{1 - 4M_\pi^2/s}\; t_0^0(s) = 0$$

$\Rightarrow$ Cubic equation for s

- Pair of complex zeros, $m_\sigma = 365 - i\,291$ MeV
- Correction from higher orders amounts to

$$\Delta m_\sigma = 76\,{}^{+16}_{-8} \;+ i\,19\,{}^{+9}_{-13} \text{ MeV}$$

For the quantity that counts, the accuracy is modest

Estimate pole position on back of an envelope

- Approximate $t_0^0(s)$ with the Weinberg formula

$$t_0^0(s) = \frac{(2s - M_\pi^2)}{32\pi F_\pi^2}$$

Where are the zeros of $S_0^0(s)$ in this approximation ?

$$1 + 2\,i\,\sqrt{1 - 4M_\pi^2/s}\; t_0^0(s) = 0$$

$\Rightarrow$ Cubic equation for s

- Pair of complex zeros, $m_\sigma = 365 - i\,291$ MeV
- Correction from higher orders amounts to

$$\Delta m_\sigma = 76\,{}^{+16}_{-8} \; + i\,19\,{}^{+9}_{-13} \text{ MeV}$$

For the quantity that counts, the accuracy is modest

- Real zero on sheet II, near $s = 0$ (full amplitude has kinematic singularity: vanishes on sheet II at $s = 0$)

Conclusion

- Low energy pion physics: theory ahead of experiment
  - Precision experiments carried out and under way
  - Lattice makes slow, but steady progress
  - So far, all tests confirm the theory

Conclusion

- Low energy pion physics: theory ahead of experiment
  - Precision experiments carried out and under way
  - Lattice makes slow, but steady progress
  - So far, all tests confirm the theory
- Limitations of our approach:
  - Calculations cannot be done on back of an envelope
  - Method only covers low energies
  - Only a few applications have been worked out:
    $\pi\pi$ scattering, pion form factors, hadronic vacuum polarization in SM prediction for muon $g-2$
    $\gamma\gamma \rightarrow \pi^0\pi^0$ M. Pennington, hep-ph/0604212
- Much is yet to be done: $J/\psi \rightarrow \omega\pi\pi$, $D \rightarrow 3\pi$, ...
  πK, κ, ...

Conclusion

- Model independent method for analytic continuation
  - The lowest resonance of QCD occurs at

    $$M_\sigma = 441\,{}^{+16}_{-8}\ \text{MeV} \qquad \Gamma_\sigma = 544\,{}^{+18}_{-25}\ \text{MeV}$$

    and carries vacuum quantum numbers
  - Crossing symmetry plays an essential role:
    Fixes contributions from left hand cut
    Ensures fast convergence, low energy dominance
  - Pole occurs at low value of s, closer to left hand cut than to singularities from $K\bar{K}$, $f_0(980)$
  - Result for Γ_σ relies on theory for a_0^2
    Experiments concerning a_0^2 would be most welcome