Introduction to Chiral Perturbation Theory

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Local symmetries

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Precision theory for $E \ll 100\,\text{GeV}$

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3. Chiral symmetry
Some of the quarks happen to be light

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Spontaneous symmetry breakdown

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If N_f of the quark masses are put equal to zero

QCD contains $N_f^2 - 1$ Goldstone bosons

5. Gell-Mann-Oakes-Renner relation
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Goldstone bosons pick up mass

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Symmetry group of the ground state H

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I. Standard Model at low energies

1. Interactions

strong weak e.m. gravity

SU(3) × SU(2) × U(1) × D

Gravity

understood only at classical level
gravitational waves √
quantum theory of gravity ?
classical theory should be adequate for

$$r \gg \sqrt{\frac{G\,\hbar}{c^3}} = 1.6 \cdot 10^{-35}\,\text{m}$$

Weak interaction

frozen at low energies

$$E \ll M_{\text{W}}\, c^2 \simeq 80\,\text{GeV}$$

⇒ structure of matter: only strong and electromagnetic interaction

⇒ neutrini decouple

Electromagnetic interaction

Maxwell $\sim$ 1860

survived relativity and quantum theory, unharmed

- Electrons in electromagnetic field ($\hbar = c = 1$)

$$\frac{1}{i}\frac{\partial\psi}{\partial t} - \frac{1}{2m_e^2}(\vec{\nabla} + i\,e\vec{A})^2\psi - e\,\varphi\,\psi = 0$$

contains the potentials $\vec{A}$, φ

- only $\quad \vec{E} = -\vec{\nabla}\varphi - \dfrac{\partial\vec{A}}{\partial t} \quad$ and $\quad \vec{B} = \vec{\nabla}\times\vec{A}$

are of physical significance

- Schrödinger equation is invariant under gauge transformations

$$\vec{A}\,' = \vec{A} + \vec{\nabla}f \;\; , \quad \varphi\,' = \varphi - \frac{\partial f}{\partial t} \;\; , \quad \psi\,' = e^{-ief}\,\psi$$

describe the same physical situation as $\vec{A}, \varphi, \psi$

- Equivalence principle of the e.m. interaction:

$$\boxed{\psi \text{ physically equivalent to } e^{-ief}\,\psi}$$

- e^{-ief} is unitary 1×1 matrix, $e^{-ief} \in$ U(1)
  $f = f(\vec{x}, t)$ space-time dependent function

- gauge invariance $\Longleftrightarrow$ local U(1) symmetry
  electromagnetic field is gauge field of U(1)
  Weyl 1929

- U(1) symmetry + renormalizability
  fully determine the e.m. interaction

Strong interaction

nuclei = p + n $\sim$ 1930

- Nuclear forces

  Yukawa $\sim$ 1935

$$V_{e.m.} = -\frac{e^2}{4\pi r} \qquad\qquad V_s = -\frac{h^2}{4\pi r}\, e^{-\frac{r}{r_0}}$$

$$\frac{e^2}{4\pi} \simeq \frac{1}{137} \qquad\qquad \frac{h^2}{4\pi} \simeq 13$$

long range — short range

$$r_0 = \infty \qquad\qquad r_0 = \frac{\hbar}{M_\pi c} = 1.4 \cdot 10^{-15}\,\mathrm{m}$$

$$M_\gamma = 0 \qquad\qquad M_\pi c^2 \simeq 140\,\mathrm{MeV}$$

- Problem with Yukawa formula:

  p and n are extended objects

  diameter comparable to range of force

  formula only holds for $r \gg$ diameter

- Protons, neutrons composed of quarks

  p $= uud$ n $= udd$

- Quarks carry internal quantum number

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \qquad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

  occur in 3 “colours”

- Strong interaction is invariant under local rotations in colour space 1973

$$u' = U \cdot u \qquad d' = U \cdot d$$

$$U = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \in \text{SU(3)}$$

- Can only be so if the strong interaction is also mediated by a gauge field

  gauge field of SU(3) $\Longrightarrow$ strong interaction

  Quantum chromodynamics

Comparison of e.m. and strong interaction

	QED	QCD
symmetry	U(1)	SU(3)
gauge field	$\vec{A}, \varphi$	gluon field
particles	photons	gluons
source	charge	colour
coupling constant	e	g

- All charged particles generate e.m. field
- All coloured particles generate gluon field
- Leptons do not interact strongly because they do not carry colour
- Equivalence principle of the strong interaction:

$$U \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \text{ physically equivalent to } \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

2. QED+QCD

Effective theory for $E \ll M_{\rm W}c^2 \simeq 80\,\text{GeV}$

Symmetry $\quad$ U(1) × SU(3)

Lagrangian $\quad$ QED+QCD

- Dynamical variables:
  gauge fields for photons and gluons
  Fermi fields for leptons and quarks

- Interaction fully determined by group geometry
  Lagrangian contains 2 coupling constants

$$\boxed{e,\; g}$$

- Quark and lepton mass matrices can be brought to diagonal form, eigenvalues real, positive

$$\boxed{m_e,\; m_\mu,\; m_\tau,\; m_u,\; m_d,\; m_s,\; m_c,\; m_b,\; m_t}$$

- Transformation generates vacuum angle

$$\boxed{\theta}$$

- θ breaks CP

  Neutron dipole moment is very small

$\Rightarrow$ strong upper limit, $\theta \simeq 0$

$\Rightarrow$ Precision theory for cold matter, atomic structure, solids, ...

Bohr radius: $$a = \frac{4\pi}{e^2\, m_e}$$

Qualitative difference between e.m. and strong interactions

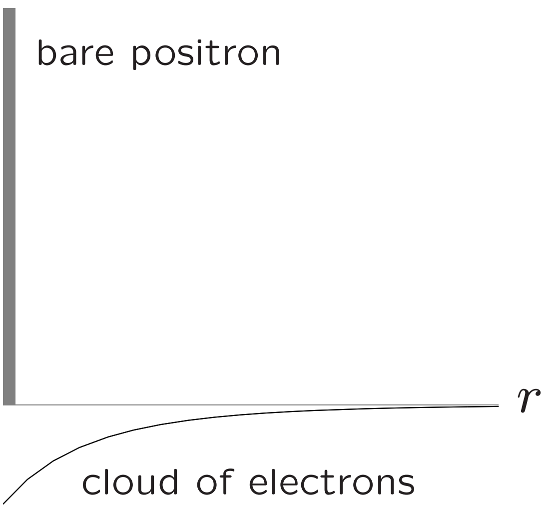
- Photons do not have charge
- Gluons do have colour

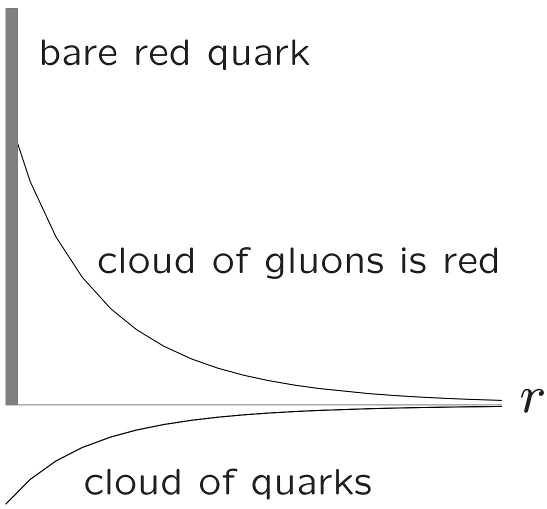
$x_1 \cdot x_2 = x_2 \cdot x_1$ for $x_1, x_2 \in U(1)$ abelian

$x_1 \cdot x_2 \neq x_2 \cdot x_1$ for $x_1, x_2 \in SU(3)$

$\Rightarrow$ Consequence for vacuum polarization

QED	QCD
Density of charge	Density of colour



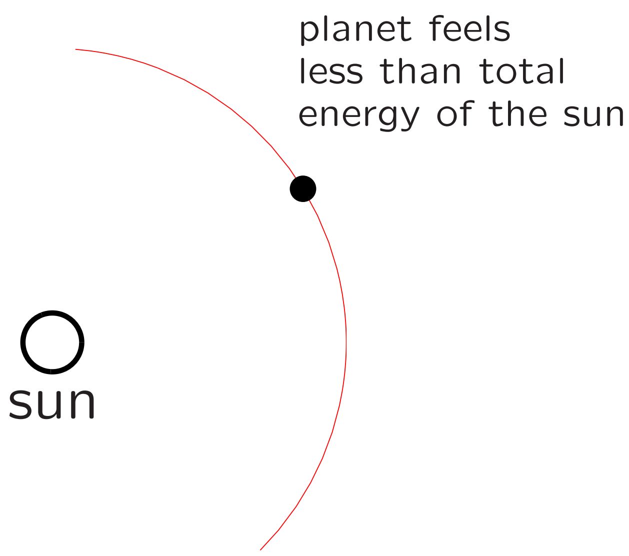


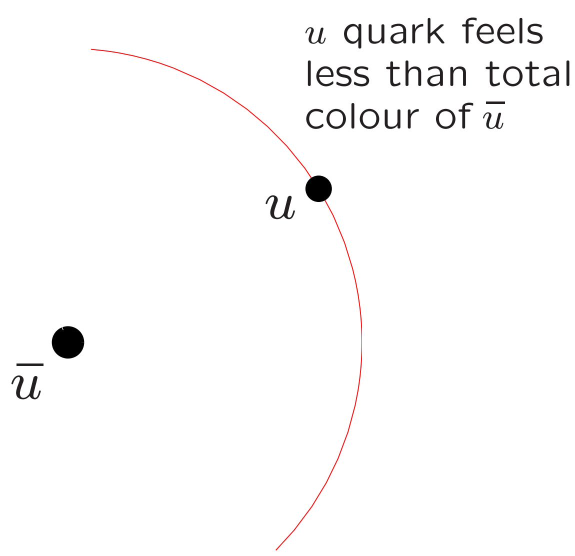
QED	QCD
$e < e_{\text{bare}}$	$g > g_{\text{bare}}$
vacuum shields charge	vacuum amplifies colour

Comparison with gravity

- source of gravitational field: energy
  gravitational field <u>does</u> carry energy
- source of e.m. field: charge
  e.m. field <u>does not</u> carry charge
- source of gluon field: colour
  gluon field <u>does</u> carry colour

gravity **strong interaction**





Perihelion shift of Mercury:

$$43'' = 50'' - 7'' \text{ per century}$$
$$\Uparrow$$

- Force between u and $\bar{u}$:

$$V_s = -\frac{4}{3}\frac{g^2}{4\pi r}\,, \qquad g \to 0 \quad \text{for} \quad r \to 0$$

$$\frac{g^2}{4\pi} = \frac{6\pi}{(11N_c - 2N_f)\,|\ln(r\,\Lambda_{\text{QCD}})|}$$

$$|\ln(r\,\Lambda_{\text{QCD}})| \simeq 7 \quad \text{for} \;\; r = \frac{\hbar}{M_Z\,c} \simeq 2\cdot 10^{-18}\,\text{m}$$

- Vacuum amplifies gluonic field of a bare quark
- Field energy surrounding isolated quark $= \infty$
  Only colour neutral states have finite energy

⇒ Confinement of colour

- Theoretical evidence for confinement meagre
  Experimental evidence much more convincing

QED: interaction weak at low energies
QCD: interaction strong at low energies

$$\frac{e^2}{4\pi} \simeq \frac{1}{137} \qquad\qquad \frac{g^2}{4\pi} \simeq 1$$

photons, leptons nearly decouple — gluons, quarks confined

- Nuclear forces = van der Waals forces of QCD

3. Chiral symmetry

- For bound states of quarks,
  e.m. interaction is a small perturbation

  Perturbation series in powers of $\frac{e^2}{4\pi}$ $\surd$

  Discuss only the leading term: set $e = 0$

- Lagrangian then reduces to QCD

$$\boxed{g\,,\,m_u\,,m_d\,,\,m_s\,,\,m_c\,,\,m_b\,,\,m_t}$$

- m_u, m_d, m_s happen to be light

  Consequence:
  Approximate flavour symmetries
  Play a crucial role for the low energy properties

Theoretical paradise

$m_u = m_d = m_s = 0$

$m_c = m_b = m_t = \infty$

QCD with 3 massless quarks

- Lagrangian contains a single parameter: g
  g is net colour of a quark
  depends on radius of the region considered

- Colour contained within radius r

$$\frac{g^2}{4\pi} = \frac{2\pi}{9\,|\ln(r\,\Lambda_{\text{QCD}})|}$$

- Intrinsic scale Λ_{QCD} is meaningful,
  but not dimensionless

$\Rightarrow$ No dimensionless free parameter

All dimensionless physical quantities are pure numbers, determined by the theory
Cross sections can be expressed in terms of Λ_{QCD} or in the mass of the proton

- Interactions of u, d, s are identical
  If the masses are set equal to zero,
  there is no difference at all

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

- Lagrangian symmetric under $u \leftrightarrow d \leftrightarrow s$

$$q' = V \cdot q \qquad V \in \mathrm{SU(3)}$$

  V acts on quark flavour, mixes u, d, s

- More symmetry: For massless fermions,
  right and left do not communicate

⇒ Lagrangian of massless QCD is invariant under independent rotations of the right– and left–handed quark fields

$$q_{\mathrm{R}} = \tfrac{1}{2}(1 + \gamma_5)\, q \,, \qquad q_{\mathrm{L}} = \tfrac{1}{2}(1 - \gamma_5)\, q$$

$$q'_{\mathrm{R}} = V_{\mathrm{R}} \cdot q_{\mathrm{R}} \qquad\qquad q'_{\mathrm{L}} = V_{\mathrm{L}} \cdot q_{\mathrm{L}}$$

$$G = \mathrm{SU(3)}_{\mathrm{R}} \times \mathrm{SU(3)}_{\mathrm{L}}$$

- Massless QCD invariant under

  $G = \mathrm{SU}(3)_{\mathrm{R}} \times \mathrm{SU}(3)_{\mathrm{L}}$

  $\mathrm{SU}(3)$ has 8 parameters

$\Rightarrow$ Symmetry under Lie group with 16 parameters

$\Rightarrow$ 16 conserved "charges"
$Q_1^{\mathrm{V}}, \ldots, Q_8^{\mathrm{V}}$ (vector currents)
$Q_1^{\mathrm{A}}, \ldots, Q_8^{\mathrm{A}}$ (axial currents)

commute with the Hamiltonian:

$$[Q_i^{\mathrm{V}}, H_0] = 0 \qquad [Q_i^{\mathrm{A}}, H_0] = 0$$

"Chiral symmetry" of massless QCD

- Vafa and Witten 1984: state of lowest energy is invariant under the vector charges
  $Q_i^{\mathrm{V}} |0\rangle = 0$

- Axial charges ? $Q_i^{\mathrm{A}} |0\rangle = ?$

Two alternatives for axial charges

$Q_i^{\mathsf{A}} \left\|0\right\rangle = 0$
Wigner-Weyl realization of G ground state is symmetric $\langle 0 \| \bar{q}_{\mathsf{R}}\, q_{\mathsf{L}} \|0\rangle = 0$ ordinary symmetry spectrum contains parity partners degenerate multiplets of G

$Q_i^{\mathsf{A}} \left\|0\right\rangle \neq 0$
Nambu-Goldstone realization of G ground state is asymmetric $\langle 0 \| \bar{q}_{\mathsf{R}}\, q_{\mathsf{L}} \|0\rangle \neq 0$ “order parameter” spontaneously broken symmetry spectrum contains Goldstone bosons degenerate multiplets of $\mathsf{SU}(3)_{\mathsf{V}} \subset \mathsf{G}$

$$\mathsf{G} = \mathsf{SU}(3)_{\mathsf{R}} \times \mathsf{SU}(3)_{\mathsf{L}}$$

- Spontaneous symmetry breakdown was discovered in condensed matter physics:

  Spontaneous magnetization selects direction

$\Rightarrow$ Rotation symmetry is spontaneously broken

  Goldstone bosons = spin waves, magnons

- Nambu 1960: state of lowest energy in particle physics is not invariant under chiral rotations

$$Q_i^{\mathsf{A}}\,|0\rangle \neq 0$$

  For dynamical reasons, the state of lowest energy must be asymmetric

$\Rightarrow$ Chiral symmetry is spontaneously broken

- Very strong experimental evidence $\surd$
- Theoretical understanding on the basis of the QCD Lagrangian ?

- Analog of Magnetization ?

$$\bar{q}_R\, q_L = \begin{pmatrix} \bar{u}_R\, u_L & \bar{d}_R\, u_L & \bar{s}_R\, u_L \\ \bar{u}_R\, d_L & \bar{d}_R\, d_L & \bar{s}_R\, d_L \\ \bar{u}_R\, s_L & \bar{d}_R\, s_L & \bar{s}_R\, s_L \end{pmatrix}$$

Transforms like $(\bar{3}, 3)$ under $SU(3)_R \times SU(3)_L$

If the ground state were symmetric, the matrix $\langle 0|\, \bar{q}_R\, q_L\, |0\rangle$ would have to vanish, because it singles out a direction in flavour space

"quark condensate", is quantitative measure of spontaneous symmetry breaking
"order parameter"

$\langle 0|\, \bar{q}_R\, q_L\, |0\rangle \Leftrightarrow$ magnetization

- Ground state is invariant under $SU(3)_V$

$\Rightarrow$ $\langle 0|\, \bar{q}_R\, q_L\, |0\rangle$ is proportional to unit matrix

$\langle 0|\, \bar{u}_R\, u_L\, |0\rangle = \langle 0|\, \bar{d}_R\, d_L\, |0\rangle = \langle 0|\, \bar{s}_R\, s_L\, |0\rangle$

$\langle 0|\, \bar{u}_R\, d_L\, |0\rangle = \ldots = 0$

4. Goldstone Theorem

- Consequence of $Q_i^{\mathsf{A}}|0\rangle \neq 0$:

  $$H_0\, Q_i^{\mathsf{A}}\,|0\rangle = Q_i^{\mathsf{A}}\, H_0\,|0\rangle = 0$$

  spectrum must contain 8 states

  $$Q_1^{\mathsf{A}}\,|0\rangle, \ldots, Q_8^{\mathsf{A}}\,|0\rangle \qquad \text{with} \quad E = 0,$$

  spin 0, negative parity, octet of $\mathsf{SU(3)_V}$
  Goldstone bosons

- Argument is not water tight:

  $$\langle 0|\, Q_i^{\mathsf{A}}\, Q_k^{\mathsf{A}}\,|0\rangle = \int d^3x d^3y\, \langle 0|\, A_i^0(x)\, A_k^0(y)\,|0\rangle$$

  $\langle 0|\, A_i^0(x)\, A_k^0(y)\,|0\rangle$ only depends on $\vec{x} - \vec{y}$

  $\Rightarrow$ $\langle 0|\, Q_i^{\mathsf{A}}\, Q_k^{\mathsf{A}}\,|0\rangle$ is proportional to the volume of the universe, $|Q_i^{\mathsf{A}}\,|0\rangle| = \infty$

- Rigorous version of Goldstone theorem:

  $\langle 0|\, \bar{q}_{\mathrm{R}}\, q_{\mathrm{L}}\, |0\rangle \neq 0 \Rightarrow \exists$ massless particles

Proof

$$Q = \int d^3x\, \bar{u}\gamma^0\gamma_5 d$$
$$[Q, \bar{d}\gamma_5 u] = -\bar{u}u - \bar{d}d$$

- $F^\mu(x-y) \equiv \langle 0|\, \bar{u}(x)\gamma^\mu\gamma_5 d(x)\, \bar{d}(y)\gamma_5 u(y)\, |0\rangle$

  Lorentz invariance $\Rightarrow$ $F^\mu(z) = z^\mu f(z^2)$
  Chiral symmetry $\Rightarrow$ $\partial_\mu F^\mu(z) = 0$

$\Rightarrow \quad 4f(z^2) + 2z^2 f'(z^2) = 0$

$\Rightarrow \quad f(z^2) = \dfrac{\text{constant}}{z^4}$

$\Rightarrow \quad F^\mu(z) = \dfrac{z^\mu}{z^4} \times \text{constant}$

- Compare Källen–Lehmann representation:

$$\langle 0|\overline{u}(x)\gamma^\mu\gamma_5 d(x)\,\bar{d}(y)\gamma_5 u(y)\,|0\rangle$$

$$= (2\pi)^{-3}\int d^4p\, p^\mu\, \rho(p^2)e^{-ip(x-y)}$$

$$= \int_0^\infty ds\,\rho(s)\partial^\mu\Delta^+(x-y,s)$$

$\Delta^+(z,s)$: propagator for particle of mass $\sqrt{s}$

$$\Delta^+(z,s) = \frac{i}{(2\pi)^3}\int d^4p\,\theta(p^0)\,\delta(p^2-s)\,e^{-ipz}$$

- Massless propagator:

$$\Delta^+(z,0) = \frac{1}{4\pi\, i\, z^2}$$

$$\Rightarrow \partial_\mu\Delta^+(z,0) = \frac{z_\mu}{z^4}\times \text{constant}$$

- Result

$$\boxed{\langle 0|\overline{u}(x)\gamma^\mu\gamma_5 d(x)\,\bar{d}(y)\gamma_5 u(y)\,|0\rangle = C\,\partial^\mu\Delta^+(z,0)}$$

$\Rightarrow$ Only massless intermediate states contribute:

$$\rho(s) = C\,\delta(s)$$

- Why only massless intermediate states ?

  $\langle n|\,\bar{d}\gamma_5 u\,|0\rangle \neq 0$ only if $\langle n|$ has spin 0

  If $|n\rangle$ has spin 0 $\Rightarrow \langle 0|\bar{u}(x)\gamma^\mu\gamma_5 d(x)|n\rangle \propto p^\mu\, e^{-ipx}$

  $\partial_\mu(\bar{u}\gamma^\mu\gamma_5 d) = 0 \Rightarrow p^2 = 0$

$\Rightarrow$ Either $\exists$ massless particles or $C = 0$

- Claim: $\langle 0|\,\bar{q}_\mathrm{R}\, q_\mathrm{L}\,|0\rangle \neq 0 \Rightarrow C \neq 0$

  Lorentz invariance, chiral symmetry

$\Rightarrow \langle 0|\,\bar{d}(y)\gamma_5 u(y)\bar{u}(x)\gamma^\mu\gamma_5 d(x)\,|0\rangle = C'\,\partial^\mu\Delta^-(z)$

$\Rightarrow \langle 0|\,[\bar{u}(x)\gamma^\mu\gamma_5 d(x),\, \bar{d}(y)\gamma_5 u(y)]\,|0\rangle$

$$= C\partial^\mu\Delta^+(z,0) - C'\partial^\mu\Delta^-(z,0)$$

- Causality: if $x - y$ is spacelike, then

  $\langle 0|\,[\bar{u}(x)\gamma^\mu\gamma_5 d(x),\, \bar{d}(y)\gamma_5 u(y)]\,|0\rangle = 0$

$\Rightarrow C' = -C$

$\Rightarrow \langle 0|\,[\bar{u}(x)\gamma^\mu\gamma_5 d(x),\, \bar{d}(y)\gamma_5 u(y)]\,|0\rangle = C\partial^\mu\Delta(z,0)$

$\Rightarrow \langle 0|\,[\,Q,\, \bar{d}(y)\gamma_5 u(y)]\,|0\rangle = C$

- $\langle 0|\,[\,Q,\, \bar{d}(y)\gamma_5 u(y)]\,|0\rangle = -\langle 0|\bar{u}u + \bar{d}d\,|0\rangle = C$

  Hence $\langle 0|\bar{u}u + \bar{d}d\,|0\rangle \neq 0$ implies $C \neq 0$ qed.

5. Gell-Mann-Oakes-Renner relation

⇒ Spectrum of QCD with 3 massless quarks must contain 8 massless physical particles, $J^P = 0^-$

- Indeed, the 8 lightest mesons do have these quantum numbers:

  $\pi^+, \pi^0, \pi^-, K^+, K^0, \bar{K}^0, K^-, \eta$

  But massless they are not

- Real world $\neq$ paradise

  $m_u\,,\, m_d\,,\, m_s \neq 0$

  Quark masses break chiral symmetry, allow left to talk to right

- Chiral symmetry broken in two ways:

  spontaneously $\langle 0|\,\overline{q}_\text{R}\, q_\text{L}\,|0\rangle \neq 0$

  explicitly $m_u\,,\, m_d\,,\, m_s \neq 0$

- H_{QCD} only has approximate symmetry to the extent that m_u, m_d, m_s are small

$$H_{\mathsf{QCD}} = H_0 + H_1$$
$$H_1 = \int d^3x \, \{m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s\}$$

- H_0 is Hamiltonian of the massless theory, invariant under $\mathsf{SU(3)_R \times SU(3)_L}$

- H_1 breaks the symmetry, transforms with $(3, \bar{3}) \oplus (\bar{3}, 3)$

- For the low energy structure of QCD, the heavy quarks do not play an essential role: c, b, t are singlets under $\mathsf{SU(3)_R \times SU(3)_L}$ Can include the heavy quarks in H_0

- Goldstone bosons are massless only if the symmetry is exact

$$\boxed{M_\pi^2 = (m_u + m_d) \times |\langle 0|\bar{u}\, u\,|0\rangle| \times \frac{1}{F_\pi^2}}$$ 1968

$$\Uparrow \qquad\qquad \Uparrow$$

explicit spontaneous

Coefficient: decay constant F_π

Derivation

- Pion matrix elements in massless theory:

$$\langle 0|\bar{u}\gamma^\mu\gamma_5 d|\pi^-\rangle = i\,\sqrt{2}\,F\,p^\mu$$
$$\langle 0|\bar{u}\,i\,\gamma_5 d|\pi^-\rangle = \sqrt{2}\,G$$

Only the one–pion intermediate state

$$\langle 0|\bar{u}(x)\gamma^\mu\gamma_5 d(x)\,\bar{d}(y)\gamma_5 u(y)\,|0\rangle = C\,\partial^\mu\Delta^+(z,0)$$

$$\Uparrow$$
$$|\pi^-\rangle\langle\pi^-|$$

contributes. Hence $2\,F\,G = C$

- Value of C fixed by quark condensate

$$C = -\langle 0|\bar{u}u + \bar{d}d\,|0\rangle$$

$\Rightarrow$ Exact result in massless theory:

$$\boxed{F\,G = -\langle 0|\bar{u}u\,|0\rangle}$$

- Matrix elements for $m_{\text{quark}} \neq 0$:

$$\langle 0|\bar{u}\gamma^\mu\gamma_5 d|\pi^-\rangle = i\sqrt{2}\,F_\pi\,p^\mu$$
$$\langle 0|\bar{u}\,i\,\gamma_5 d|\pi^-\rangle = \sqrt{2}\,G_\pi$$

- Current conservation

$$\partial_\mu(\bar{u}\gamma^\mu\gamma_5 d) = (m_u + m_d)\bar{u}\,i\,\gamma_5 d$$
$$\Rightarrow\ F_\pi M_\pi^2 = (m_u + m_d)\,G_\pi$$

$$\boxed{M_\pi^2 = (m_u + m_d)\,\frac{G_\pi}{F_\pi}} \qquad \text{exact for } m \neq 0$$

- $F_\pi \to F\,, \quad G_\pi \to G \quad$ for $m \to 0$

$$F\,G = -\langle 0|\bar{u}u\,|0\rangle$$

$$\Rightarrow \frac{G_\pi}{F_\pi} = -\frac{\langle 0|\bar{u}u\,|0\rangle}{F_\pi^2} + O(m)$$

$$\Rightarrow M_\pi^2 = (m_u + m_d)\left(\frac{-\langle 0|\bar{u}u\,|0\rangle}{F_\pi^2}\right) + O(m^2)\ \surd$$

$\Rightarrow \langle 0|\bar{u}u\,|0\rangle \leq 0$ if quark masses are positive

- $M_\pi^2 = (m_u + m_d)\, B + O(m^2)$

  $$B = \frac{|\langle 0|\bar{u}\, u\,|0\rangle|}{F_\pi^2}$$

- M_π disappears if the symmetry breaking is turned off, $m_u, m_d \rightarrow 0$ $\surd$

- Explains why the pseudoscalar mesons have very different masses

  $M_{K^+}^2 = (m_u + m_s)\, B + O(m^2)$

  $M_{K^-}^2 = (m_d + m_s)\, B + O(m^2)$

$\Rightarrow$ M_K^2 is about 13 times larger than M_π^2, because m_u, m_d happen to be small compared to m_s

- First order perturbation theory also yields

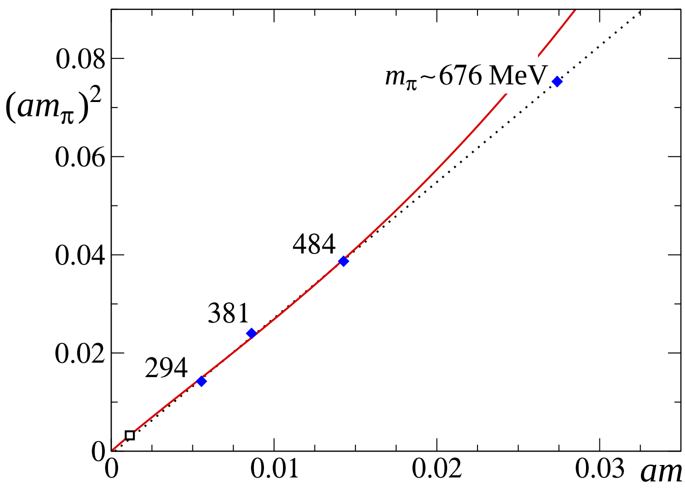
  $M_\eta^2 = \frac{1}{3}(m_u + m_d + 4m_s)B + O(m^2)$

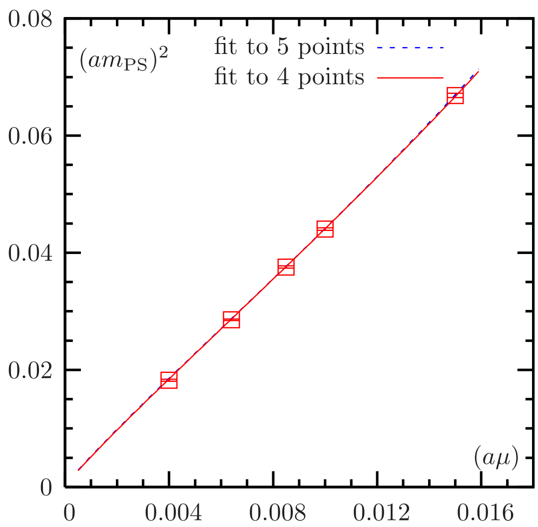
$\Rightarrow$ $M_\pi^2 - 4M_K^2 + 3M_\eta^2 = O(m^2)$

Gell-Mann-Okubo formula for M^2 $\surd$

Checking the GMOR formula on a lattice

- Can determine M_π as function of $m_u = m_d = m$





Lüscher, Lattice conference 2005 ETM collaboration, hep-lat/0701012

- No quenching, quark masses sufficiently light

⇒ Legitimate to use χPT for the extrapolation to the physical values of m_u, m_d

- Quality of data is impressive
- Proportionality of M_π^2 to the quark mass appears to hold out to values of m_u, m_d that are an order of magnitude larger than in nature
- Main limitation: systematic uncertainties

  in particular: $N_f = 2 \to N_f = 3$

II. Chiral perturbation theory

6. Group geometry

- QCD with 3 massless quarks:
  spontaneous symmetry breakdown
  from $\text{SU}(3)_\text{R}\times\text{SU}(3)_\text{L}$ to $\text{SU}(3)_\text{V}$
  generates 8 Goldstone bosons

- Generalization: suppose symmetry group
  of Hamiltonian is Lie group G
  Generators $Q_1, Q_2, \ldots, Q_D$, $D = \text{dim(G)}$
  For some generators $Q_i\,|0\rangle \neq 0$
  How many Goldstone bosons ?

- Consider those elements of the Lie algebra
  $Q = \alpha_1 Q_1 + \ldots + \alpha_n Q_D$, for which $Q\,|0\rangle = 0$
  These elements form a subalgebra:
  $Q\,|0\rangle = 0$, $Q'\,|0\rangle = 0 \;\Rightarrow [Q, Q']\,|0\rangle = 0$
  Dimension of subalgebra: $d \leq D$

- Of the D vectors $Q_i\,|0\rangle$
  $D - d$ are linearly independent

$\Rightarrow$ $D - d$ different physical states of zero mass
$\Rightarrow$ $D - d$ Goldstone bosons

- Subalgebra generates subgroup H $\subset$ G
  H is symmetry group of the ground state
  coset space G/H contains as many parameters as there are Goldstone bosons
  $d = \text{dim(H)}$, $D = \text{dim(G)}$

$\Rightarrow$ Goldstone bosons live on the coset G/H

- Example: QCD with N_f massless quarks
  G $= \text{SU}(N_f)_\text{R} \times \text{SU}(N_f)_\text{L}$
  H $= \text{SU}(N_f)_\text{V}$
  $D = 2\,(N_f^2 - 1)$, $d = N_f^2 - 1$
  $N_f^2 - 1$ Goldstone bosons

- It so happens that $m_u, m_d \ll m_s$

- $m_u = m_d = 0$ is an excellent approximation
  SU(2)$_\text{R}\times$ SU(2)$_\text{L}$ is a nearly exact symmetry
  $N_f = 2$, $N_f^2 - 1 = 3$ Goldstone bosons (pions)

7. Generating functional

- Basic objects for quantitative analysis of QCD: Green functions of the currents

$$V_a^\mu = \bar{q}\,\gamma^\mu \tfrac{1}{2}\lambda_a\, q\,, \;\; A_a^\mu = \bar{q}\,\gamma^\mu\gamma_5 \tfrac{1}{2}\lambda_a\, q\,,$$
$$S_a = \bar{q}\,\tfrac{1}{2}\lambda_a\, q\,, \qquad P_a = \bar{q}\,i\,\gamma_5\,\tfrac{1}{2}\lambda_a\, q$$

Include singlets, with $\lambda_0 = \sqrt{2/3}\times \mathbf{1}$, as well as

$$\omega = \frac{1}{16\pi^2}\,\mathrm{tr}_c\, G_{\mu\nu}\tilde{G}^{\mu\nu}$$

- Can collect all of the Green functions formed with these operators in a generating functional: Perturb the system with external fields
$v_\mu^a(x)$, $a_\mu^a(x)$, $s_a(x)$, $p^a(x)$, $\theta(x)$
Replace the Lagrangian of the massless theory

$$\mathcal{L}_0 = -\frac{1}{2g^2}\,\mathrm{tr}_c\, G_{\mu\nu}G^{\mu\nu} + \bar{q}\,i\gamma^\mu(\partial_\mu - i\,G_\mu)\,q$$

by $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$

$$\mathcal{L}_1 = v_\mu^a V_a^\mu + a_\mu^a A_a^\mu - s^a S_a - p^a P_a - \theta\,\omega$$

- Quark mass terms are included in the external field $s_a(x)$

- $|0\,\text{in}\rangle$: system is in ground state for $x^0 \to -\infty$
  Probability amplitude for finding ground state when $x^0 \to +\infty$:

$$e^{iS_{\text{QCD}}\{v,a,s,p,\theta\}} = \langle 0\,\text{out}|0\,\text{in}\rangle_{v,a,s,p,\theta}$$

  $S_{\text{QCD}}\{v,a,s,p,\theta\}$ "generating functional of QCD"

- Expressed in terms of ground state of $\mathcal{L}_0$:

$$e^{iS_{\text{QCD}}\{v,a,s,p,\theta\}} = \langle 0|\, T \exp i \int dx \mathcal{L}_1 \,|0\rangle$$

- Expansion of $S_{\text{QCD}}\{v,a,s,p,\theta\}$ in powers of the external fields yields the connected parts of the Green functions of the massless theory

$$S_{\text{QCD}}\{v,a,s,p,\theta\} = -\int dx\, s_a(x) \langle 0|\, S^a(x)\,|0\rangle$$
$$+\tfrac{i}{2}\int dx dy\, a^a_\mu(x) a^b_\nu(y) \langle 0|\, T A^\mu_a(x) A^\nu_b(y)\,|0\rangle_{\text{conn}} + \ldots$$

- For Green functions of full QCD, set

$$s_a(x) = m_a + \tilde{s}_a(x)\,, \qquad m_a = \text{tr}\lambda_a\, m$$

  and expand around $\tilde{s}_a(x) = 0$

- Path integral representation of generating functional:

$$\boxed{e^{iS_{\text{QCD}}\{v,a,s,p\}} = \mathcal{N}\int [dG]\, e^{\,i\int dx\, \mathcal{L}_{\text{G}}}\ \det D}$$

$$\mathcal{L}_{\text{G}} = -\frac{1}{2g^2}\,\underset{c}{\text{tr}}\, G_{\mu\nu}G^{\mu\nu} - \frac{\theta}{16\pi^2}\,\underset{c}{\text{tr}}\, G_{\mu\nu}\tilde{G}^{\mu\nu}$$

$$D = i\gamma^\mu\{\partial_\mu - i(G_\mu + v_\mu + a_\mu\gamma_5)\} - s - i\gamma_5 p$$

G_μ is matrix in colour space

v_μ, a_μ, s, p are matrices in flavour space

$v_\mu(x) \equiv \frac{1}{2}\lambda_a\, v^a_\mu(x)$, etc.

8. Ward identities

Symmetry in terms of Green functions

- Lagrangian is invariant under

$$q_{\mathrm{R}}(x) \to V_{\mathrm{R}}(x)\, q_{\mathrm{R}}(x)\,, \qquad q_{\mathrm{L}}(x) \to V_{\mathrm{L}}(x)\, q_{\mathrm{L}}(x)$$
$$V_{\mathrm{R}}(x), V_{\mathrm{L}}(x) \in \mathrm{U}(3)$$

provided the external fields are transformed with

$$v'_\mu + a'_\mu = V_{\mathrm{R}}(v_\mu + a_\mu)V_{\mathrm{R}}^\dagger - i\partial_\mu V_{\mathrm{R}} V_{\mathrm{R}}^\dagger$$
$$v'_\mu - a'_\mu = V_{\mathrm{L}}(v_\mu - a_\mu)V_{\mathrm{L}}^\dagger - i\partial_\mu V_{\mathrm{L}} V_{\mathrm{L}}^\dagger$$
$$s' + i\,p' = V_{\mathrm{R}}(s + i\,p)V_{\mathrm{L}}^\dagger$$

The operation takes the Dirac operator into

$$D' = \left\{P_- V_{\mathrm{R}} + P_+ V_{\mathrm{L}}\right\} D \left\{P_+ V_{\mathrm{R}}^\dagger + P_- V_{\mathrm{L}}^\dagger\right\}$$
$$P_\pm = \tfrac{1}{2}(1 \pm \gamma_5)$$

- $\det D$ requires regularization

$\nexists$ symmetric regularization

$\Rightarrow$ $\det D' \neq \det D$, only $|\det D'| = |\det D|$

symmetry does not survive quantization

- Change in $\det D$ can explicitly be calculated

  For an infinitesimal transformation

  $V_{\mathsf{R}} = \mathbf{1} + i\,\alpha + i\beta + \ldots\,, \quad V_{\mathsf{L}} = \mathbf{1} + i\,\alpha - i\beta + \ldots$

  the change in the determinant is given by

$$\det D' = \det D \; e^{-i\int dx\,\{2\langle\beta\rangle\omega + \langle\beta\Omega\rangle\}}$$

$$\langle A\rangle \equiv \mathrm{tr}\, A$$

$$\omega = \frac{1}{16\pi^2}\,\mathrm{tr}_c\, G_{\mu\nu}\tilde{G}^{\mu\nu} \qquad \text{gluons}$$

$$\Omega = \frac{N_c}{4\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\partial_\mu v_\nu \partial_\rho v_\sigma + \ldots \qquad \text{ext. fields}$$

- Consequence for generating functional:
  The term with ω amounts to a change in θ,
  can be compensated by $\theta' = \theta - 2\,\langle\beta\rangle$
  Pull term with $\langle\beta\Omega\rangle$ outside the path integral

$$\Rightarrow \boxed{S_{\mathsf{QCD}}\{v',a',s',p',\theta'\} = S_{\mathsf{QCD}}\{v,a,s,p,\theta\} - \int dx \langle\beta\Omega\rangle}$$

$$\boxed{S_{\text{QCD}}\{v',a',s',p',\theta'\} = S_{\text{QCD}}\{v,a,s,p,\theta\} - \int dx \langle \beta\Omega\rangle}$$

- S_{QCD} is invariant under $\text{U}(3)_{\text{R}}\times\text{U}(3)_{\text{L}}$, except for a specific change due to the anomalies

- Relation plays key role in low energy analysis: collects all of the Ward identities
  For the octet part of the axial current,e.g.

$$\partial^x_\mu \langle 0|\, T A^\mu_a(x)\, P_b(y)\,|0\rangle = -\tfrac{1}{4}\, i\, \delta(x-y) \langle 0|\, \overline{q}\{\lambda_a,\lambda_b\} q\,|0\rangle$$
$$+ \langle 0|\, T\overline{q}(x)\, i\gamma_5 \{m, \tfrac{1}{2}\lambda_a\} q(x)\, P_b(y)\,|0\rangle$$

- Symmetry of the generating functional implies the operator relations

$$\partial_\mu V^\mu_a = \overline{q}\, i[m,\tfrac{1}{2}\lambda_a] q\,, \qquad a = 0,\ldots,8$$
$$\partial_\mu A^\mu_a = \overline{q}\, i\gamma_5 \{m,\tfrac{1}{2}\lambda_a\} q\,, \qquad a = 1,\ldots,8$$
$$\partial_\mu A^\mu_0 = \sqrt{\tfrac{2}{3}}\,\overline{q}\, i\gamma_5 m q + \sqrt{6}\,\omega$$

- Textbook derivation of the Ward identities goes in inverse direction, but is slippery formal manipulations, anomalies ?

9. Low energy expansion

- If the spectrum has an energy gap

⇒ no singularities in scattering amplitudes or Green functions near $p = 0$

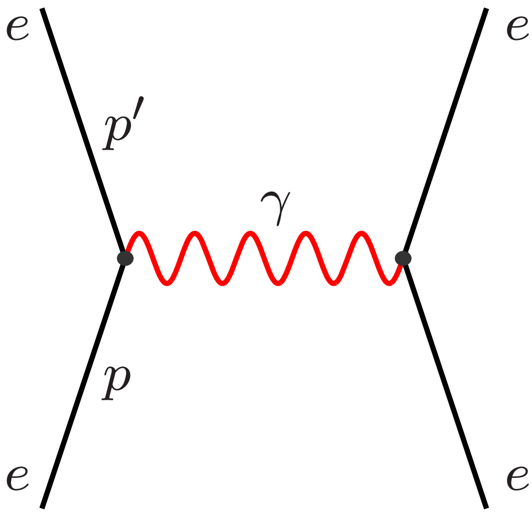
⇒ low energy behaviour may be analyzed with Taylor series expansion in powers of p

$$f(t) = 1 + \tfrac{1}{6}\langle r^2 \rangle\, t + \ldots \quad \text{form factor}$$
$$T(p) = a + b\, p^2 + \ldots \quad \text{scattering amplitude}$$

Cross section dominated by S–wave scattering length $\qquad \dfrac{d\sigma}{d\Omega} \simeq |a|^2$

- Expansion parameter: $\dfrac{p}{m} = \dfrac{\text{momentum}}{\text{energy gap}}$
- Taylor series only works if the spectrum has an energy gap, i.e. if there are <u>no massless particles</u>

- Illustration: Coulomb scattering



Photon exchange $\Rightarrow$ pole at $t = 0$

$$T = \frac{e^2}{(p' - p)^2}$$

Scattering amplitude does not admit
Taylor series expansion in powers of p

- QCD does have an energy gap
but the gap is very small: M_π

$\Rightarrow$ Taylor series has very small radius of convergence, useful only for $p < M_\pi$

- Massless QCD contains infrared singularities due to the Goldstone bosons
- For $m_u = m_d = 0$, pion exchange gives rise to poles and branch points at $p = 0$

⇒ Low energy expansion is not a Taylor series, contains logarithms

Singularities due to Goldstone bosons can be worked out with an effective field theory "Chiral Perturbation Theory"

Weinberg, Dashen, Pagels, Gasser, . . .

- Chiral perturbation theory correctly reproduces the infrared singularities of QCD
- Quantities of interest are expanded in powers of external momenta and quark masses
- Expansion has been worked out to next-to-leading order for many quantities "Chiral perturbation theory to one loop"
- In quite a few cases, the next-to-next-to-leading order is also known

- Properties of the Goldstone bosons are governed by the hidden symmetry that is responsible for their occurrence

- Focus on the singularities due to the pions
  Include the mass term of strange quark in H_0

$$H_{\text{QCD}} = H_0 + H_1$$
$$H_1 = \int d^3x \, \{m_u \bar{u}u + m_d \bar{d}d\}$$

  H_0 is invariant under $\mathrm{G} = \mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{SU}(2)_{\mathrm{L}}$

  $|0\rangle$ is invariant under $\mathrm{H} = \mathrm{SU}(2)_{\mathrm{V}}$

- Treat H_1 as a perturbation

Expansion in powers of H_1	$\Longleftrightarrow$	Expansion in powers of m_u, m_d

- Extension to $\mathrm{SU}(3)_{\mathrm{R}} \times \mathrm{SU}(3)_{\mathrm{L}}$ straightforward: include singularities due to exchange of K, η

10. Effective Lagrangian

- Replace quarks and gluons by pions

$$\vec{\pi}(x) = \{\pi^1(x), \pi^2(x), \pi^3(x)\}$$

$$\mathcal{L}_{\text{QCD}} \;\rightarrow\; \mathcal{L}_{\mathit{eff}}$$

- Central claim:

  A. Effective theory yields alternative representation for generating functional of QCD

$$\boxed{e^{iS_{\text{QCD}}\{v,a,s,p,\theta\}} = \mathcal{N}_{\mathit{eff}} \int [d\pi] e^{i\int dx \mathcal{L}_{\mathit{eff}}\{\vec{\pi},v,a,s,p,\theta\}}}$$

  B. $\mathcal{L}_{\mathit{eff}}$ has the same symmetries as $\mathcal{L}_{\text{QCD}}$

$\Rightarrow$ Can calculate the low energy expansion of the Green functions with the effective theory.
If $\mathcal{L}_{\mathit{eff}}$ is chosen properly, this reproduces the low energy expansion of QCD, order by order.

- Proof of A and B: H.L., Annals Phys. 1994

- Pions live on the coset G/H = SU(2)

$$\vec{\pi}(x) \rightarrow U(x) \in \text{SU}(2)$$

The fields $\vec{\pi}(x)$ are the coordinates of $U(x)$

Can use canonical coordinates, for instance

$$U = \exp i\,\vec{\pi}\cdot\vec{\tau} \;\in\; \text{SU}(2)$$

- Action of the symmetry group on the quarks:

$$q'_{\text{R}} = V_{\text{R}} \cdot q_{\text{R}}\,, \qquad q'_{\text{L}} = V_{\text{L}} \cdot q_{\text{L}}$$

- Action on the pion field:

$$U' = V_{\text{R}} \cdot U \cdot V_{\text{L}}^{\dagger}$$

Note: Transformation law for the coordinates $\vec{\pi}$ is complicated, nonlinear

- Except for the contribution from the anomalies, $\mathcal{L}_{\mathit{eff}}$ is invariant

$$\boxed{\mathcal{L}_{\mathit{eff}}\{U', v', a', s', p', \theta'\} = \mathcal{L}_{\mathit{eff}}\{U, v, a, s, p, \theta\}}$$

Symmetry of S_{QCD} implies symmetry of $\mathcal{L}_{\mathit{eff}}$

Side remark

- For nonrelativistic effective theories, the effective Lagrangian is in general invariant only up to a total derivative.

⇒ From the point of view of effective field theory, nonrelativistic systems with Goldstone bosons are more complicated than relativistic ones

detailed discussion: H. L., Phys. Rev. D49 (1994) 3033

- Origin of the complication: the generators of the symmetry group may themselves give rise to order parameters

$$\langle 0| Q^i |0\rangle \neq 0$$

This cannot happen in the relativistic case:

$$Q = \int d^3x \, j^0(x)$$

$$\langle 0| j^\mu(x) |0\rangle = 0 \ \Rightarrow \langle 0| Q |0\rangle = 0$$

Nonrelativistic example where it does happen:

Heisenberg model of a ferromagnet

$$H = -g \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j$$

$g > 0$ $\quad \Uparrow\Uparrow$ lower in energy than $\Uparrow\Downarrow$

- Ground state = $\Uparrow\Uparrow\Uparrow\Uparrow \cdots \Uparrow\Uparrow$
- Magnetization: $\vec{M} = \frac{\mu}{V} \sum_i \vec{s}_i$

$$\langle 0 | \vec{M} | 0 \rangle \neq 0 \iff \langle 0 | \bar{q}_R \, q_L | 0 \rangle \neq 0$$

- Symmetry generators: $\vec{Q} = \sum_i \vec{s}_i \propto \vec{M}$
- Hamiltonian is invariant under the full rotation group G = SO(3), ground state is invariant only under rotations around the direction of $\langle 0 | \vec{M} | 0 \rangle$, H = U(1)
- Effective field lives on G/H = S_2: unit vector $\vec{U}$, parametrized by 2 coordinates π^1, π^2.
- Effective Lagrangian of ferromagnet is invariant under local rotations only up to a total derivative. Leading term is related to the Brouwer degree of the map $(\pi^1, \pi^2) \to \vec{U}$.

11. Explicit construction of $\mathcal{L}_{eff}$

- First ignore the external fields,

$$\mathcal{L}_{eff} = \mathcal{L}_{eff}(U, \partial U, \partial^2 U, \ldots)$$

Derivative expansion:

$$\mathcal{L}_{eff} = f_0(U) + f_1(U) \times \Box U + f_2(U) \times \partial_\mu U \times \partial^\mu U + \ldots$$

$$\Uparrow \qquad \Uparrow \qquad \Uparrow$$

$$O(1) \qquad O(p^2) \qquad O(p^2)$$

Amounts to expansion in powers of momenta

- Term of $O(1)$: $f_0(U) = f_0(V_\mathrm{R} U V_\mathrm{L}^\dagger)$
$V_\mathrm{R} = \mathbf{1}\,, \quad V_\mathrm{L} = U \rightarrow V_\mathrm{R} U V_\mathrm{L}^\dagger = \mathbf{1}$

$\Rightarrow$ $f_0(U) = f_0(\mathbf{1})$ irrelevant constant, drop it

- Term with $\Box U$: integrate by parts

$\Rightarrow$ can absorb $f_1(U)$ in $f_2(U)$

⇒ Derivative expansion of $\mathcal{L}_{eff}$ starts with

$$\mathcal{L}_{eff} = f_2(U) \times \partial_\mu U \times \partial^\mu U + O(p^4)$$

- Replace the partial derivative by

$$\Delta_\mu \equiv \partial_\mu U U^\dagger \,, \quad \mathrm{tr}\Delta_\mu = 0$$

Δ_μ is invariant under $\mathrm{SU(2)_L}$ and transforms with the representation $D^{(1)}$ under $\mathrm{SU(2)_R}$: $\Delta_\mu \to V_\mathrm{R}\, \Delta_\mu\, V_\mathrm{R}^\dagger$

In this notation, leading term is of the form

$$\mathcal{L}_{eff} = \tilde{f}_2(U) \times \Delta_\mu \times \Delta^\mu + O(p^4)$$

- Invariance under $\mathrm{SU(2)_L}$: $\tilde{f}_2(U) = \tilde{f}_2(U V_\mathrm{L}^\dagger)$

⇒ $\tilde{f}_2(U)$ is independent of U

- Invariance under $\mathrm{SU(2)_R}$: $\Delta_\mu \times \Delta^\mu$ transforms with $D^{(1)} \times D^{(1)} \to$ contains unity exactly once: $\mathrm{tr}(\Delta_\mu \Delta^\mu) = \mathrm{tr}(\partial_\mu U U^\dagger \partial^\mu U U^\dagger) = -\mathrm{tr}(\partial_\mu U \partial^\mu U^\dagger)$

⇒ Geometry fixes leading term up to a constant

$$\boxed{\mathcal{L}_{eff} = \frac{F^2}{4}\mathrm{tr}(\partial_\mu U \partial^\mu U^\dagger) + O(p^4)}$$

$$\boxed{\mathcal{L}_{eff} = \frac{F^2}{4}\text{tr}(\partial_\mu U \partial^\mu U^\dagger) + O(p^4)}$$

- Lagrangian of the nonlinear σ-model
- Expansion in powers of $\vec{\pi}$:

$$U = \exp i\,\vec{\pi}\cdot\vec{\tau} = \mathbf{1} + i\,\vec{\pi}\cdot\vec{\tau} - \tfrac{1}{2}\,\vec{\pi}^{\,2} + \ldots$$

$\Rightarrow$ $\mathcal{L}_{eff} = \frac{F^2}{2}\,\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} + \frac{F^2}{48}\text{tr}\{[\partial_\mu\pi,\pi]\,[\partial^\mu\pi,\pi]\} + \ldots$

For the kinetic term to have the standard normalization: rescale the pion field, $\vec{\pi} \to \vec{\pi}/F$

$\mathcal{L}_{eff} = \frac{1}{2}\,\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} + \frac{1}{48F^2}\text{tr}\{[\partial_\mu\pi,\pi]\,[\partial^\mu\pi,\pi]\} + \ldots$

$\Rightarrow$ a. Symmetry requires the pions to interact

b. Derivative coupling: Goldstone bosons only interact if their momentum does not vanish $\not{\lambda}\pi^4$

- Expression given for $\mathcal{L}_{eff}$ only holds if the external fields are turned off. Also, tr$(\partial_\mu U \partial^\mu U^\dagger)$ is invariant only under global transformations

  Suffices to replace $\partial_\mu U$ by

$$D_\mu U = \partial_\mu U - i(v_\mu + a_\mu)U + i\,U(v_\mu - a_\mu)$$

  In contrast to tr$(\partial_\mu U \partial^\mu U^\dagger)$, the term tr$(D_\mu U D^\mu U^\dagger)$ is invariant under local SU(2)$_{\mathsf{R}}\times$ SU(2)$_{\mathsf{L}}$

- Can construct further invariants: $s + ip$ transforms like $U \Rightarrow$ tr$\{(s + ip)U^\dagger\}$ is invariant
  Violates parity, but tr$\{(s+ip)U^\dagger\}+$tr$\{(s-ip)U\}$ is even under $p \to -p, \vec{\pi} \to -\vec{\pi}$

  In addition, $\exists$ invariant independent of U:
  $D_\mu \theta D^\mu \theta$, with $D_\mu \theta = \partial_\mu \theta + 2\,\text{tr}(a_\mu)$

- Count the external fields as
  $\theta = O(1), \quad v_\mu, a_\mu = O(p), \quad s, p = O(p^2)$

- Derivative expansion yields string of the form

$$\mathcal{L}_{eff} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \dots$$

- Full expression for leading term:

$$\boxed{\mathcal{L}^{(2)} = \frac{F^2}{4}\langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + U\chi^\dagger\rangle + h_0 D_\mu\theta D^\mu\theta}$$

$$\chi \equiv 2\,B\,(s+ip)\,, \qquad \langle X\rangle \equiv \mathrm{tr}(X)$$

- Contains 3 constants: F, B, h_0
  "effective coupling constants"

- Next-to-leading order:

$$\begin{aligned}\mathcal{L}^{(4)} = & \frac{\ell_1}{4}\langle D_\mu U D^\mu U\rangle^2 + \frac{\ell_2}{4}\langle D_\mu U D_\nu U\rangle\langle D^\mu U D^\nu U\rangle \\ & + \frac{\ell_3}{4}\langle \chi U^\dagger + U\chi^\dagger\rangle^2 + \frac{\ell_4}{4}\langle D_\mu\chi D^\mu U^\dagger + D_\mu U D^\mu \chi^\dagger\rangle \\ & + \dots\end{aligned}$$

- Number of effective coupling constants rapidly grows with the order of the expansion

- Infinitely many effective coupling constants
  Symmetry does not determine these
  Predictivity ?

- Essential point: If $\mathcal{L}_{\mathit{eff}}$ is known to given order $\Rightarrow$ can work out low energy expansion of the Green functions to that order (Weinberg 1979)

- NLO expressions for F_π, M_π involve 2 new coupling constants: ℓ_3, ℓ_4.

  In the $\pi\pi$ scattering amplitude, two further coupling constants enter at NLO: ℓ_1, ℓ_2.

- Note: effective theory is a quantum field theory
  Need to perform the path integral

$$\boxed{e^{iS_{\text{QCD}}\{v,a,s,p,\theta\}} = \mathcal{N}_{\mathit{eff}} \int [d\pi] e^{i\int dx \mathcal{L}_{\mathit{eff}}\{\vec{\pi},v,a,s,p,\theta\}}}$$

- Classical theory $\Leftrightarrow$ tree graphs
  Need to include graphs with loops

- Power counting in dimensional regularization:
  Graphs with ℓ loops are suppressed by factor $p^{2\ell}$ as compared to tree graphs

$\Rightarrow$ Leading contributions given by tree graphs
Graphs with one loop contribute at next-to-leading order, etc.

- The leading contribution to S_{QCD} is given by the sum of all tree graphs = classical action:

$$S_{\text{QCD}}\{v,a,s,p,\theta\} = \underset{U(x)}{\text{extremum}} \int dx\, \mathcal{L}_{\mathit{eff}}\{U,v,a,s,p,\theta\}$$

$\Rightarrow$ current algebra in compact form

III. Illustrations

12. Some tree level calculations

A. Condensate in terms of QCD

- To calculate the quark condensate of the massless theory, it suffices to consider the generating functional for $v = a = p = \theta = 0$ and to take a constant scalar external field

$$s = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

- Expansion in powers of m_u and m_d treats

  $H_1 = \int d^3x \, \{m_u \bar{u}u + m_d \bar{d}d\}$ as a perturbation

  $$S_{\text{QCD}}\{0, 0, m, 0, 0\} = S^0_{\text{QCD}} + S^1_{\text{QCD}} + \ldots$$

- S^0_{QCD} is independent of the quark masses (cosmological constant)
- S^1_{QCD} is linear in the quark masses

- First order in m_u, m_d $\Rightarrow$ expectation value of H_1 in unperturbed ground state is relevant

$$S^1_{\text{QCD}} = -\int dx \langle 0|\, m_u \bar{u}u + m_d \bar{d}d \,|0\rangle$$

$\Rightarrow$ $\langle 0|\bar{u}u\,|0\rangle$ and $\langle 0|\bar{d}d\,|0\rangle$ are the coefficients of the terms in S_{QCD} that are linear in m_u and m_d

B. Condensate in terms of effective theory

- Need the generating functional for $v = a = p = \theta = 0$ to first order in s

$\Rightarrow$ classical level of effective theory suffices.

- extremum of the classical action: $U = \mathbf{1}$

$$S^1_{\text{QCD}} = \int dx F^2 B(m_u + m_d)$$

- comparison with

$$S^1_{\text{QCD}} = -\int dx \langle 0|\, m_u \bar{u}u + m_d \bar{d}d \,|0\rangle \quad \text{yields}$$

$$\boxed{\langle 0|\bar{u}u\,|0\rangle = \langle 0|\bar{d}d\,|0\rangle = -F^2 B} \qquad (1)$$

C. Evaluation of M_π at tree level

- In classical theory, the square of the mass is the coefficient of the term in the Lagrangian that is quadratic in the meson field:

$$\frac{F^2}{4}\langle \chi U^\dagger + U\chi^\dagger\rangle = \frac{F^2 B}{2}\langle m(U^\dagger + U)\rangle$$
$$= F^2 B(m_u + m_d)\{1 - \frac{\vec{\pi}^{\,2}}{2F^2} + \ldots\}$$

Hence $\boxed{M_\pi^2 = (m_u + m_d)B}$ (2)

- Tree level result for F_π:

$$\boxed{F_\pi = F} \qquad (3)$$

- (1) + (2) + (3) $\Rightarrow$ GMOR relation:

$$\boxed{M_\pi^2 = \frac{(m_u + m_d)\,|\langle 0|\,\bar{u}u\,|0\rangle|}{F_\pi^2}}$$

13. M_π beyond tree level

- The formula $M_\pi^2 = (m_u + m_d)B$ only holds at tree level, represents leading term in expansion of M_π^2 in powers of m_u, m_d
- Disregard isospin breaking: set $m_u = m_d = m$

A. M_π to 1 loop

- Claim: at next-to-leading order, the expansion of M_π^2 in powers of m contains a logarithm:

$$M_\pi^2 = M^2 - \frac{1}{2}\frac{M^4}{(4\pi F)^2}\ln\frac{\Lambda_3^2}{M^2} + O(M^6)$$
$$M^2 \equiv 2mB$$

- Proof: Pion mass $\Leftrightarrow$ pole position, for instance in the Fourier transform of $\langle 0|\, T A_a^\mu(x) A_b^\nu(y)\, |0\rangle$

  Suffices to work out the perturbation series for this object to one loop of the effective theory

- Result

$$M_\pi^2 = M^2 + \frac{2\,\ell_3 M^4}{F^2} + \frac{M^2}{2F^2}\,\frac{1}{i}\Delta(0,M^2) + O(M^6)$$

$\Delta(0,M^2)$ is the propagator at the origin

$$\Delta(0,M^2) = \frac{1}{(2\pi)^d}\int \frac{d^dp}{M^2 - p^2 - i\epsilon}$$
$$= i\,(4\pi)^{-d/2}\,\Gamma(1-d/2)M^{d-2}$$

- Contains a pole at $d=4$:

$$\Gamma\,(1-d/2) = \frac{2}{d-4} + \ldots$$

- Divergent part is proportional to M^2:

$$M^{d-2} = M^2\mu^{d-4}(M/\mu)^{d-4} = M^2\mu^{d-4}e^{(d-4)\ln(M/\mu)}$$
$$= M^2\mu^{d-4}\{1 + (d-4)\ln(M/\mu) + \ldots\}$$

- Denote the singular factor by

$$\lambda \equiv \frac{1}{2}(4\pi)^{-d/2}\,\Gamma(1-d/2)\,\mu^{d-4}$$
$$= \frac{\mu^{d-4}}{16\pi^2}\left\{\frac{1}{d-4} - \frac{1}{2}(\ln 4\pi + \Gamma'(1) + 1) + O(d-4)\right\}$$

- The propagator at the origin then becomes

$$\frac{1}{i}\Delta(0,M^2)=M^2\left\{2\lambda+\frac{1}{16\pi^2}\ln\frac{M^2}{\mu^2}+O(d-4)\right\}$$

- In the expression for M_π^2

$$M_\pi^2=M^2+\frac{2\,\ell_3 M^4}{F^2}+\frac{M^2}{2F^2}\frac{1}{i}\Delta(0,M^2)+O(M^6)$$

the divergence can be absorbed in ℓ_3:

$$\ell_3=-\frac{1}{2}\lambda+\ell_3^{\text{ren}}$$

- ℓ_3^{ren} depends on the renormalization scale μ

$$\ell_3^{\text{ren}}=\frac{1}{64\pi^2}\ln\frac{\mu^2}{\Lambda_3^2}$$ running coupling constant

- Λ_3 is the ren. group invariant scale of ℓ_3

Net result for M_π^2

$$\boxed{M_\pi^2=M^2-\frac{1}{2}\frac{M^4}{(4\pi F)^2}\ln\frac{\Lambda_3^2}{M^2}+O(M^6)}$$

$\Rightarrow$ M_π^2 contains a chiral logarithm at NLO

- Crude estimate for Λ_3, based on SU(3) mass formulae for the pseudoscalar octet:

$$0.2 \text{ GeV} < \Lambda_3 < 2 \text{ GeV}$$

$$\bar{\ell}_3 \equiv \ln \frac{\Lambda_3^2}{M_\pi^2} = 2.9 \pm 2.4$$

Gasser & L. 1984

⇒ Next–to–leading term is small correction:

$$0.005 < \frac{1}{2} \frac{M_\pi^2}{(4\pi F_\pi)^2} \ln \frac{\Lambda_3^2}{M_\pi^2} < 0.04$$

- Scale of the expansion is set by size of pion mass in units of decay constant:

$$\frac{M^2}{(4\pi F)^2} \simeq \frac{M_\pi^2}{(4\pi F_\pi)^2} = 0.0144$$

B. M_π to 2 loops

- Terms of order m_{quark}^3:

$$M_\pi^2 = M^2 - \frac{1}{2}\,\frac{M^4}{(4\pi F)^2}\,\ln\frac{\Lambda_3^2}{M^2} + \frac{17}{18}\,\frac{M^6}{(4\pi F)^4}\left(\ln\frac{\Lambda_{\text{M}}^2}{M^2}\right)^2 + k_{\text{M}}M^6 + O(M^8)$$

F is pion decay constant for $m_u = m_d = 0$
ChPT to two loops Colangelo 1995

- Coefficients $\frac{1}{2}$ and $\frac{17}{18}$ determined by symmetry
- $\Lambda_3, \Lambda_{\text{M}}$ and $k_{\text{M}} \Longleftrightarrow$ coupling constants in $\mathcal{L}_{\mathit{eff}}$

14. F_π to one loop

- Also contains a logarithm at NLO:

$$F_\pi = F\left\{1 - \frac{M^2}{16\pi^2 F^2}\ln\frac{M^2}{\Lambda_4^2} + O(M^4)\right\}$$
$$M_\pi^2 = M^2\left\{1 + \frac{M^2}{32\pi^2 F^2}\ln\frac{M^2}{\Lambda_3^2} + O(M^4)\right\}$$

F is pion decay constant in limit $m_u, m_d \to 0$

- Structure is the same, coefficients and scale of logarithm are different

- Low energy theorem: Λ_4 also determines the slope of the scalar form factor to leading order

$$\langle r^2 \rangle_s = \frac{6}{(4\pi F)^2} \left\{ \ln \frac{\Lambda_4^2}{M_\pi^2} - \frac{13}{12} + O(M^2) \right\}$$

- Scalar form factor of the pion can be calculated by means of dispersion theory
- Result for the slope:

$$\langle r^2 \rangle_s = 0.61 \pm 0.04 \, \text{fm}^2$$

Colangelo, Gasser & L. Nucl. Phys. 2001

⇒ Corresponding value of the scale Λ_4:

$$\Lambda_4 = 1.26 \pm 0.14 \, \text{GeV}$$

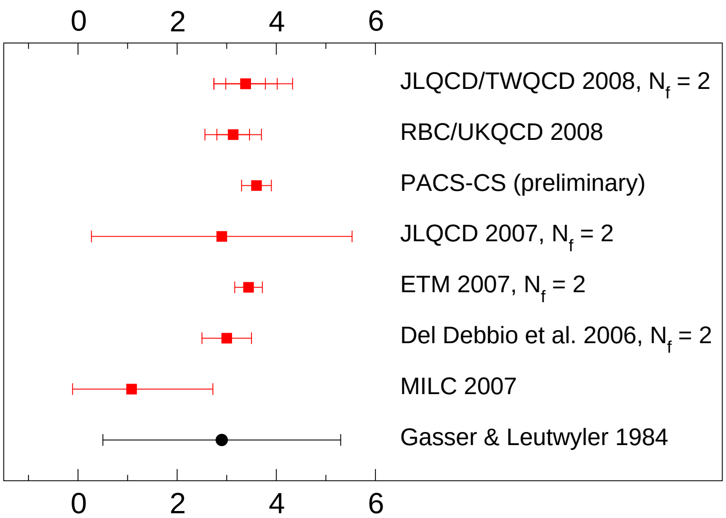
15. Lattice results for M_π, F_π

A. Results for M_π

- Determine the scale Λ_3 by comparing the lattice results for M_π as function of m with the χPT formula

$$M_\pi^2 = M^2 - \frac{1}{2}\frac{M^4}{(4\pi F)^2}\ln\frac{\Lambda_3^2}{M^2} + O(M^6)$$

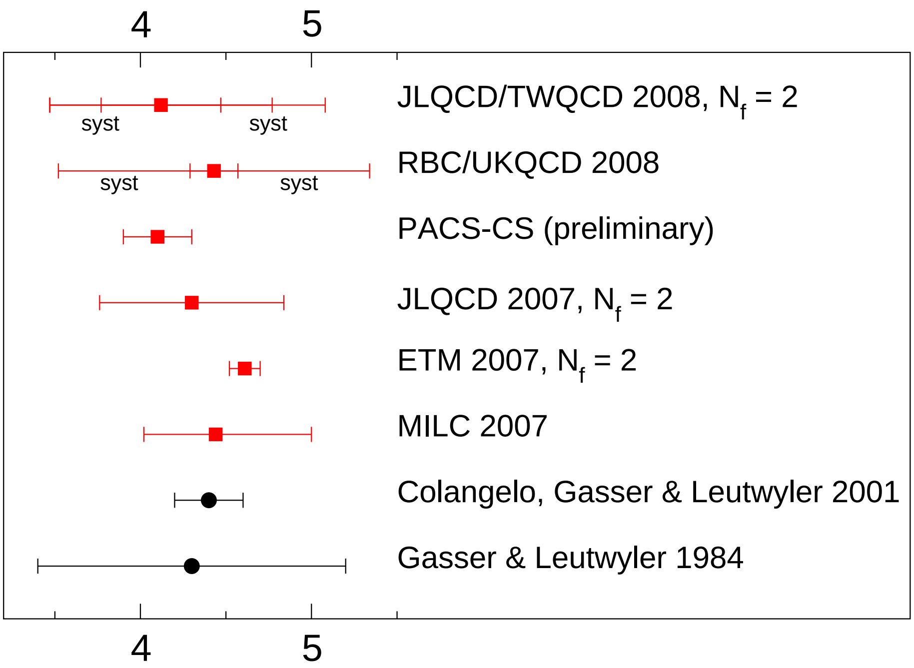
$$M^2 \equiv 2Bm$$



Horizontal axis shows the value of $\bar{\ell}_3 \equiv \ln\frac{\Lambda_3^2}{M_\pi^2}$

Range for Λ_3 obtained in 1984 corresponds to $\bar{\ell}_3 = 2.9 \pm 2.4$

Result of RBC/UKQCD 2008: $\bar{\ell}_3 = 3.13 \underset{stat}{\pm 0.33} \underset{syst}{\pm 0.24}$



$$\bar{\ell}_4 = \ln \frac{\Lambda_4^2}{M_\pi^2}$$

- Lattice results beautifully confirm the prediction for the sensitivity of F_π to m_u, m_d:

$$\frac{F_\pi}{F} = 1.072 \pm 0.007 \qquad \text{Colangelo and Dürr 2004}$$

16. $\pi\pi$ scattering

A. Low energy scattering of pions

- Consider scattering of pions with $\vec{p} = 0$
- At $\vec{p} = 0$, only the S-waves survive (angular momentum barrier). Moreover, these reduce to the scattering lengths
- Bose statistics: S-waves cannot have $I = 1$, either have $I = 0$ or $I = 2$

⇒ At $\vec{p} = 0$, the $\pi\pi$ scattering amplitude is characterized by two constants: a_0^0, a_0^2

- Chiral symmetry suppresses the interaction at low energy: Goldstone bosons of zero momentum do not interact

⇒ a_0^0, a_0^2 disappear in the limit $m_u, m_d \to 0$

⇒ $a_0^0, a_0^2 \sim M_\pi^2$ measure symmetry breaking

B. Tree level of χPT

- Low Energy theorem Weinberg 1966:

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} + O(M_\pi^4)$$
$$a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} + O(M_\pi^4)$$

⇒ Chiral symmetry predicts a_0^0, a_0^2 in terms of F_π

- Accuracy is limited: Low energy theorem only specifies the first term in the expansion in powers of the quark masses
  Corrections from higher orders ?

C. Scattering lengths at 1 loop

- Next term in the chiral perturbation series:

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2}\left\{1 + \frac{9}{2}\,\frac{M_\pi^2}{(4\pi F_\pi)^2}\,\ln\frac{\Lambda_0^2}{M_\pi^2} + O(M_\pi^4)\right\}$$

- Coefficient of chiral logarithm unusually large
  Strong, attractive final state interaction
- Scale Λ_0 is determined by the coupling constants of $\mathcal{L}_{eff}^{(4)}$:

$$\frac{9}{2}\ln\frac{\Lambda_0^2}{M_\pi^2} = \frac{20}{21}\,\bar{\ell}_1 + \frac{40}{21}\,\bar{\ell}_2 - \frac{5}{14}\,\bar{\ell}_3 + 2\,\bar{\ell}_4 + \frac{5}{2}$$

- Information about $\bar{\ell}_1, \ldots, \bar{\ell}_4$?

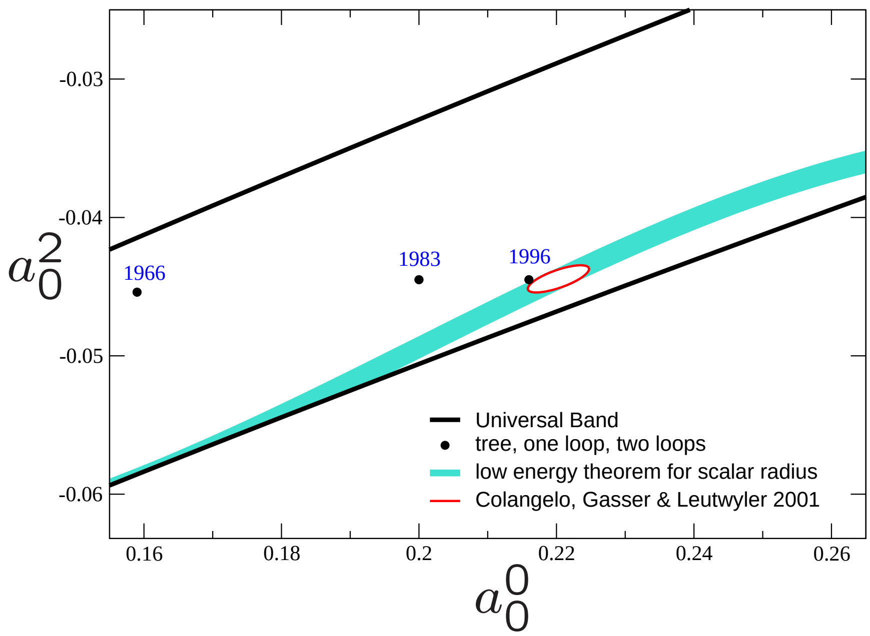
$\bar{\ell}_1, \bar{\ell}_2$	$\Longleftrightarrow$	momentum dependence of scattering amplitude

$\Rightarrow$ Can be determined phenomenologically

$\bar{\ell}_3, \bar{\ell}_4$	$\Longleftrightarrow$	dependence of scattering amplitude on quark masses

Have discussed their values already

D. Numerical predictions from χPT



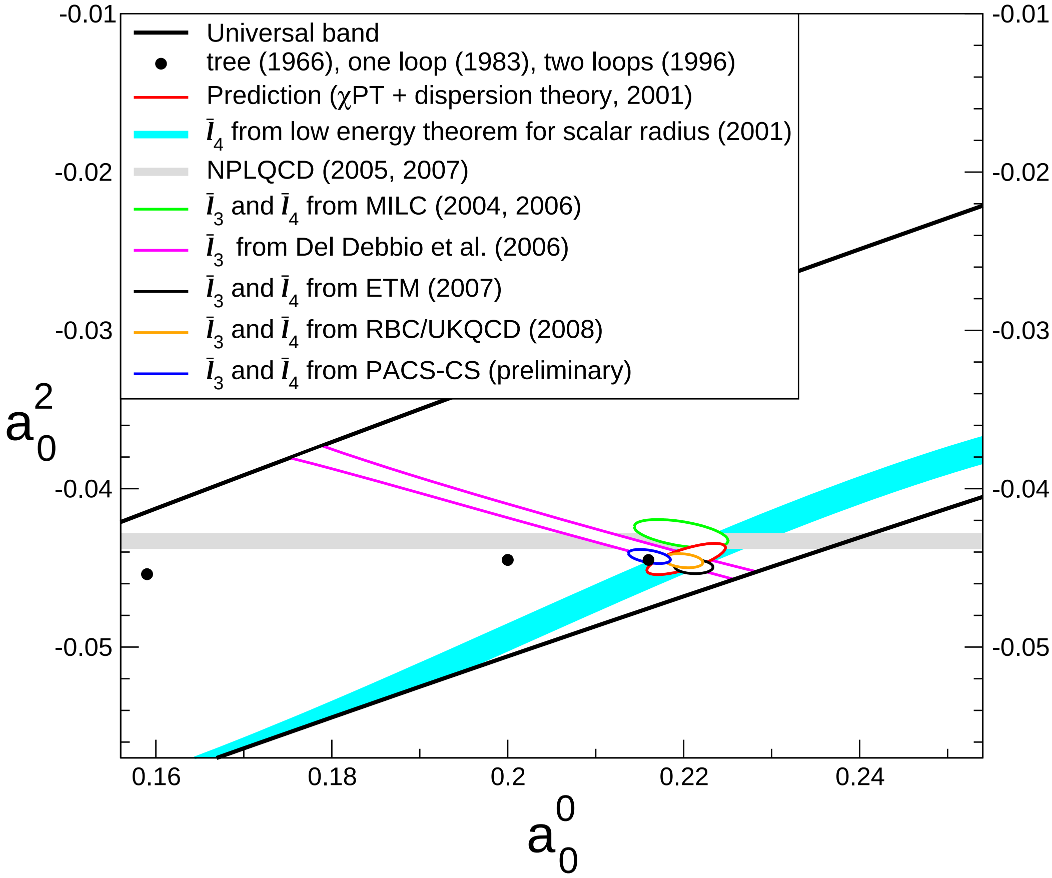
Sizable corrections in a_0^0

a_0^2 nearly stays put

E. Consequence of lattice results for ℓ_3, ℓ_4

- Uncertainty in prediction for a_0^0, a_0^2 is dominated by the uncertainty in the effective coupling constants ℓ_3, ℓ_4

- Can make use of the lattice results for these



F. Experiments concerning a_0^0, a_0^2

- Production experiments $\pi N \to \pi\pi N$,
  $\psi \to \pi\pi\omega$, $B \to D\pi\pi$, . . .

  Problem: pions are not produced in vacuo

$\Rightarrow$ Extraction of $\pi\pi$ scattering amplitude is not simple

  Accuracy rather limited

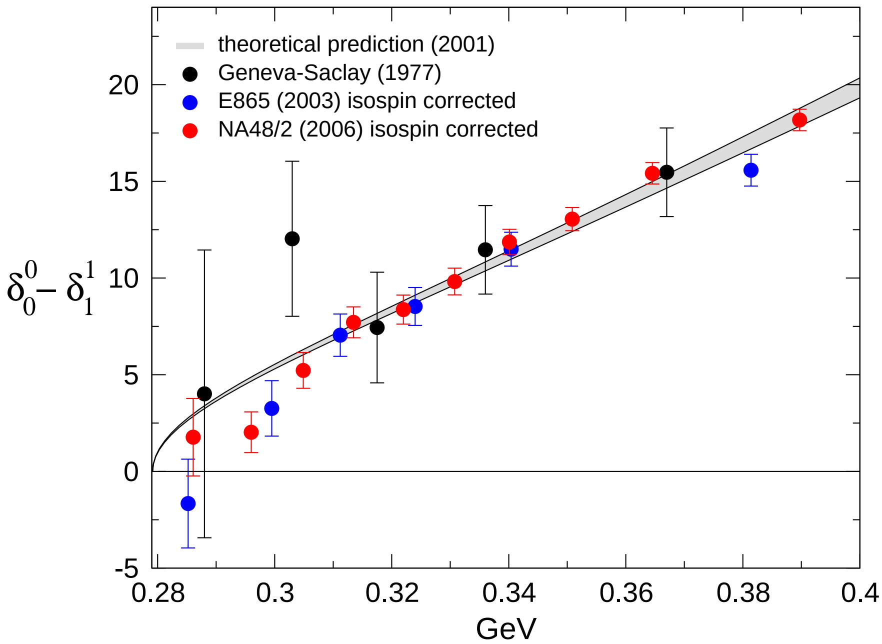
- $K^\pm \to \pi^+\pi^- e^\pm \nu$ data:
  CERN-Saclay, E865, NA48/2
- $K^\pm \to \pi^0\pi^0\pi^\pm$, $K^0 \to \pi^0\pi^0\pi^0$: cusp near threshold, NA48/2
- $\pi^+\pi^-$ atoms, DIRAC

G. Results from K_{e4} decay

$K^{\pm} \to \pi^{+}\pi^{-}e^{\pm}\nu$

- Allows clean measurement of $\delta_0^0 - \delta_1^1$

Theory predicts $\delta_0^0 - \delta_1^1$ as function of energy



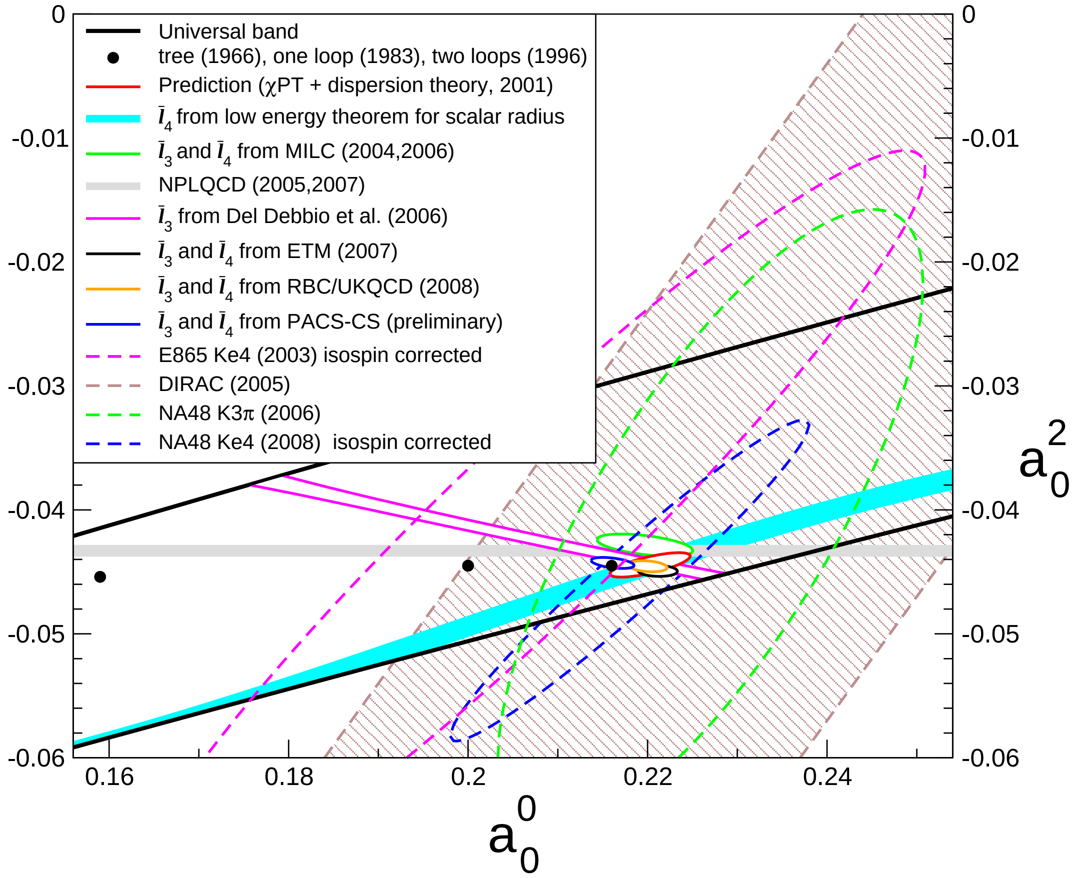
There was a discrepancy here, because a pronounced isospin breaking effect from

$$K \to \pi^0\pi^0 e\nu \to \pi^+\pi^- e\nu$$

had not been accounted for in the data analysis

$\Rightarrow$ Colangelo, Gasser, Rusetsky, arXiv:0811.0775

H. Summary for a_0^0, a_0^2



17. Conclusions for SU(2)×SU(2)

- Expansion in powers of m_u, m_d yields a very accurate low energy representation of QCD
- Lattice results confirm the GMOR relation

⇒ M_π is dominated by the contribution from the quark condensate

⇒ Energy gap of QCD is understood very well

- Lattice approach allows an accurate measurement of the effective coupling constant ℓ_3 already now
- Even for ℓ_4, the lattice starts becoming competitive with dispersion theory

18. Expansion in powers of m_s

- Theoretical reasoning
  - The eightfold way is an approximate symmetry
  - The only coherent way to understand this within QCD: $m_s - m_d$, $m_d - m_u$ can be treated as perturbations
  - Since $m_u, m_d \ll m_s$

  $\Rightarrow$ m_s can be treated as a perturbation

  $\Rightarrow$ Expect expansion in powers of m_s to work, but convergence to be comparatively slow

- This can now also be checked on the lattice

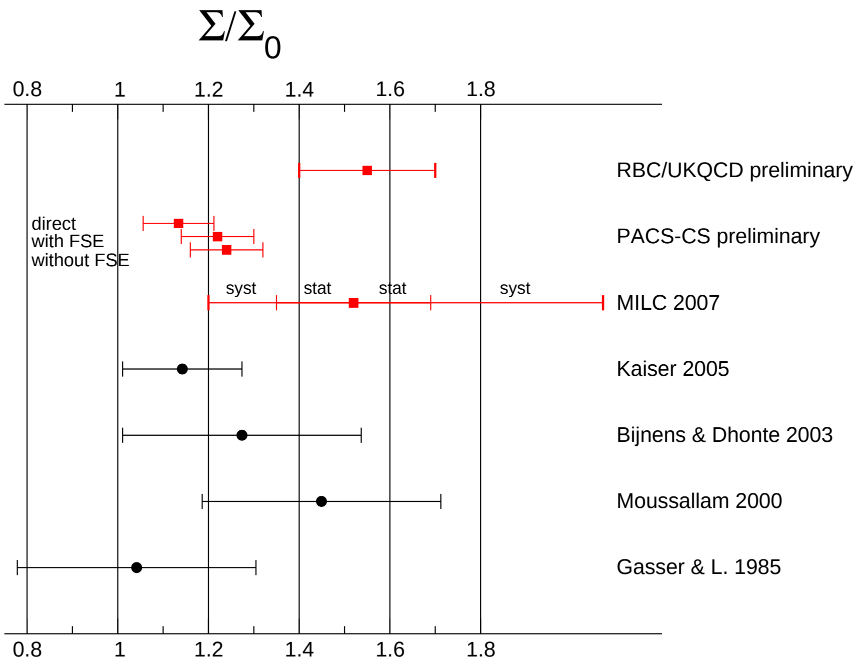
- Consider the limit $m_u, m_d \to 0$, m_s physical
    - F is value of F_π in this limit
    - Σ is value of $|\langle 0|\,\bar{u}u\,|0\rangle|$ in this limit
    - B is value of $M_\pi^2/(m_u+m_d)$ in this limit
- Exact relation: $\Sigma = F^2 B$
- F_0, B_0, Σ_0: values for $m_u = m_d = m_s = 0$
- *Paramagnetic inequalities*: both F and Σ should decrease if m_s is taken smaller

  $F > F_0\,,\ \Sigma > \Sigma_0$ Jan Stern et al. 2000

- $N_c \to \infty$: F, Σ, B become independent of m_s

$\Rightarrow$ $(F/F_0-1), (\Sigma/\Sigma_0-1), (B/B_0-1)$
violate the OZI rule

A. Condensate

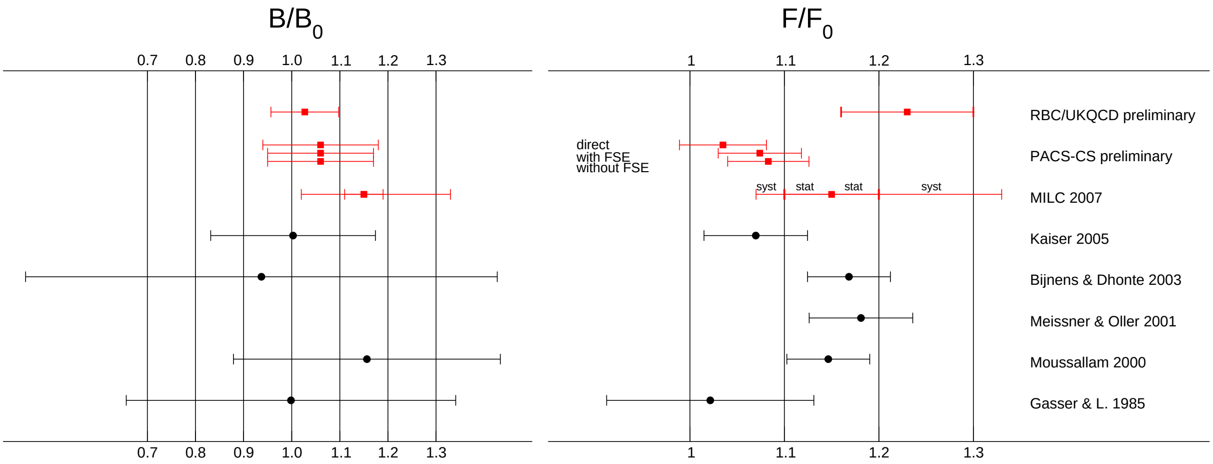


- Central values of RBC/UKQCD and PACS-CS for Σ/Σ_0 lead to qualitatively different conclusions concerning OZI-violations

$\Rightarrow$ Discrepancy indicates large systematic errors

- The lattice results confirm the parametric inequalities, but do not yet allow to draw conclusions about the size of the OZI-violations

B. Results for B, F



- Results for B are coherent, indicate small OZI-violations in B

$\Rightarrow$ F is the crucial factor in $\Sigma = F^2 B$

- Note: most of the numbers quoted are preliminary, errors purely statistical, continuum limit, finite size effects, . . .

C. Expansion to NLO

Involves the effective coupling constants L_4 and L_6 of the SU(3)×SU(3) Lagrangian:

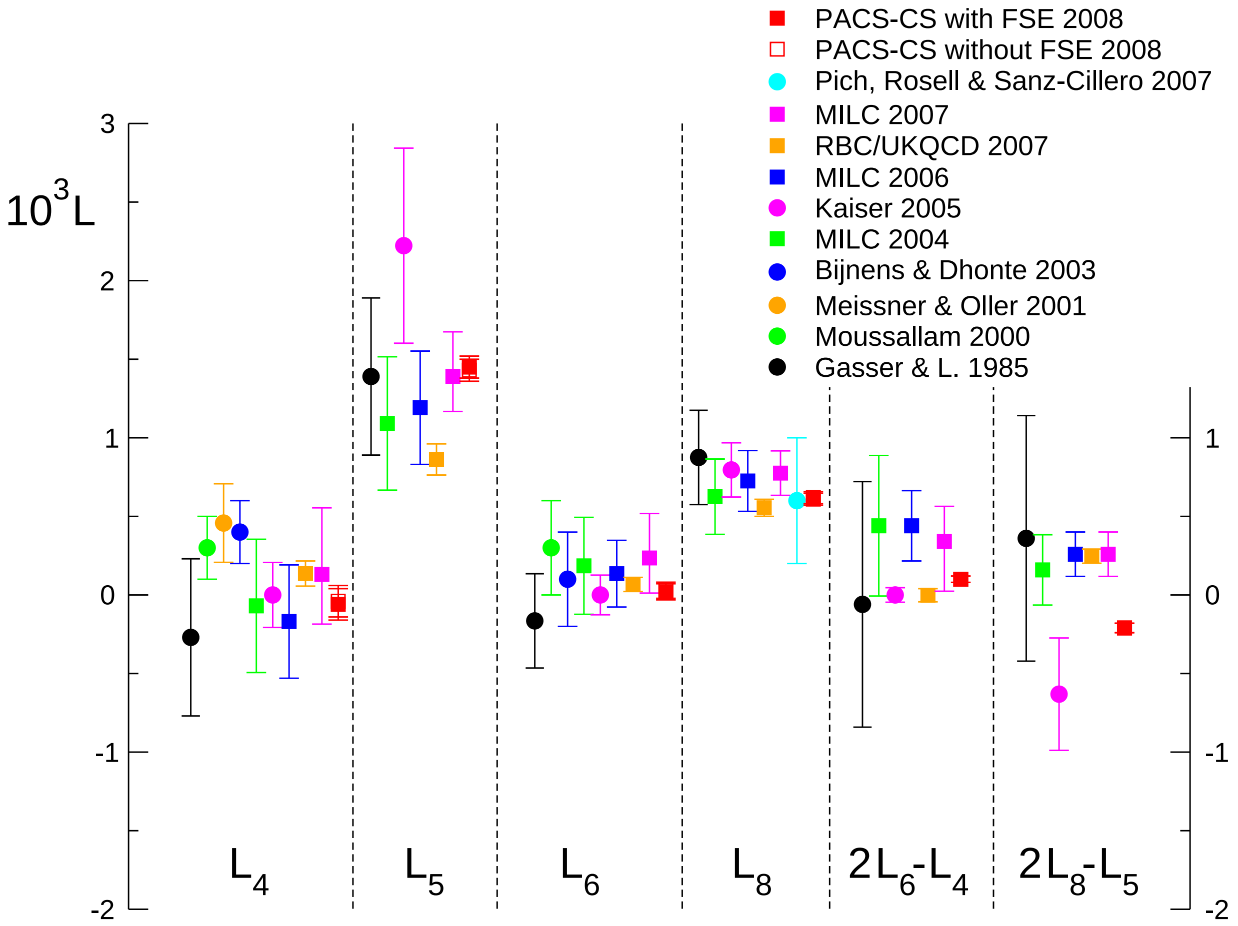
$$F/F_0 = 1 + \frac{8\,\overline{M}_K^2}{F_0^2}\,L_4 + \chi\text{log} + \ldots$$

$$\Sigma/\Sigma_0 = 1 + \frac{32\,\overline{M}_K^2}{F_0^2}\,L_6 + \chi\text{log} + \ldots$$

$$B/B_0 = 1 + \frac{16\,\overline{M}_K^2}{F_0^2}\,(2L_6 - L_4) + \chi\text{log} + \ldots$$

$\overline{M}_K$ is the kaon mass for $m_u = m_d = 0$.

D. Running coupling constants L_4, L_5, L_6, L_8



Numerical values shown refer to running scale $\mu = M_\rho$

19. Conclusions for SU(3)×SU(3)

- In $B/B_0 \leftrightarrow 2L_6-L_4$, the available lattice results indicate little if any violations of the OZI rule
- For F/F_0 (and hence Σ/Σ_0), the situation is not conclusive: some of the data indicate very juicy OZI-violations, others are consistent with $F/F_0 \simeq \Sigma/\Sigma_0 \simeq 1$
- If the central value $F/F_0 = 1.23$ of RBC/UKQCD were confirmed within small uncertainties, we would be faced with a qualitative puzzle:
  - F_π is the pion wave function at the origin
  - F_K is larger because one of the two valence quarks is heavier $\rightarrow$ moves more slowly $\rightarrow$ wave function more narrow $\rightarrow$ higher at the origin: $F_K/F_\pi \simeq 1.19$
  - $F/F_0 = 1.23$ indicates that the wave function is more sensitive to the mass of the sea quarks than to the mass of the valence quarks ... very strange $\rightarrow$ most interesting if true

- The PACS-CS results are consistent with our old estimates. Only show modest violations of the OZI rule. If these results are confirmed, then the picture looks very coherent, also for SU(3)$\times$SU(3).

Exercises

1. Evaluate the positive frequency part of the massless propagator

$$\Delta^{+}(z,0) = \frac{i}{(2\pi)^3} \int \frac{d^3k}{2k^0}\, e^{-ikz} \quad , \qquad k^0 = |\vec{k}|$$

for $\mathrm{Im}\, z^0 < 0$. Show that the result can be represented as

$$\Delta^{+}(z,0) = \frac{1}{4\pi i z^2}$$

2. Evaluate the d-dimensional propagator

$$\Delta(z,M) = \int \frac{d^dk}{(2\pi)^d} \frac{e^{-ikz}}{M^2 - k^2 - i\epsilon}$$

at the origin and verify the representation

$$\Delta(0,M) = \frac{i}{4\pi}\Gamma\left(1 - \frac{d}{2}\right)\left(\frac{M^2}{4\pi}\right)^{\frac{d}{2}-1}$$

How does this expression behave when $d \to 4$?

3. Leading order effective Lagrangian:

$$
\begin{aligned}
\mathcal{L}^{(2)} &= \frac{F^2}{4}\langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + U\chi^\dagger\rangle + h_0 D_\mu\theta D^\mu\theta \\
D_\mu U &= \partial_\mu U - i(v_\mu + a_\mu)U + i\,U(v_\mu - a_\mu) \\
\chi &= 2\,B\,(s+ip) \\
D_\mu\theta &= \partial_\mu\theta + 2\langle a_\mu\rangle \\
\langle X\rangle &= \mathrm{tr}X
\end{aligned}
$$

- Take the space-time independent part of the external field $s(x)$ to be isospin symmetric (i. e. set $m_u = m_d = m$):

$$s(x) = m\,\mathbf{1} + \tilde{s}(x)$$

- Expand $U = \exp i\,\phi/F$ in powers of $\phi = \vec{\phi}\cdot\vec{\tau}$ and check that, in this normalization of the field ϕ, the kinetic part takes the standard form

$$\mathcal{L}^{(2)} = \tfrac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \tfrac{1}{2}M^2\vec{\phi}^2 + \ldots$$

  with $M^2 = 2mB$.

- Draw the graphs for all of the interaction vertices containing up to four of the fields $\phi, v_\mu, a_\mu, \tilde{s}, p, \theta$.

4. Show that the classical field theory belonging to the QCD Lagrangian in the presence of external fields is invariant under

$$\begin{aligned} v'_\mu + a'_\mu &= V_{\mathrm{R}}(v_\mu + a_\mu)V_{\mathrm{R}}^\dagger - i\partial_\mu V_{\mathrm{R}} V_{\mathrm{R}}^\dagger \\ v'_\mu - a'_\mu &= V_{\mathrm{L}}(v_\mu - a_\mu)V_{\mathrm{L}}^\dagger - i\partial_\mu V_{\mathrm{L}} V_{\mathrm{L}}^\dagger \\ s' + i\,p' &= V_{\mathrm{R}}(s + i\,p)V_{\mathrm{L}}^\dagger \\ q'_{\mathrm{R}} &= V_{\mathrm{R}}\, q_{\mathrm{R}}(x) \\ q'_{\mathrm{L}} &= V_{\mathrm{L}}\, q_{\mathrm{L}} \end{aligned}$$

where $V_{\mathrm{R}}, V_{\mathrm{L}}$ are space-time dependent elements of U(3).

5. Evaluate the pion mass to NLO of χPT. Draw the relevant graphs and verify the representation

$$M_\pi^2 = M^2 + \frac{2\,\ell_3 M^4}{F^2} + \frac{M^2}{2F^2}\frac{1}{i}\Delta(0, M^2) + O(M^6)$$

6. Start from the symmetry property of the generating functional,

$$S_{\mathrm{QCD}}\{v', a', s', p', \theta'\} = S_{\mathrm{QCD}}\{v, a, s, p, \theta\} - \int dx \langle \beta\Omega\rangle,$$

and show that this relation in particular implies the Ward identity

$$\begin{aligned} \partial^x_\mu \langle 0|\, T A^\mu_a(x)\, P_b(y)\,|0\rangle &= -\tfrac{1}{4}\, i\,\delta(x-y)\langle 0|\,\overline{q}\{\lambda_a, \lambda_b\}q\,|0\rangle \\ &\quad + \langle 0|\, T\overline{q}(x)\, i\gamma_5\{m, \tfrac{1}{2}\lambda_a\}q(x)\, P_b(y)\,|0\rangle \\ a = 1,\ldots,8,&\ \ b = 0,\ldots,8 \end{aligned}$$

7. What is the Ward identity obeyed by the singlet axial current,

$$\partial^x_\mu \langle 0|\, T A^\mu_0(x)\, P_b(y)\,|0\rangle = \ ?$$